Worksheet 3: Products

Definition. The **direct product** of the groups $(G_1, \star_1)$ and $(G_2, \star_2)$ is the group

$$G_1 \times G_2 = \{(x, y) \mid x \in G_1, y \in G_2\}$$

where the binary operation is $(x, y) \star (z, w) = (x \star_1 z, y \star_2 w)$.
The groups G_1 and G_2 are called the **factors** of G.

1. Practice with direct product groups.

   (a) List the elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, and then list the subgroup $\langle (x, y) \rangle$ generated by each element $(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. What is the order of every element in $\mathbb{Z}_2 \times \mathbb{Z}_2$? Is $\mathbb{Z}_2 \times \mathbb{Z}_2$ cyclic?

   (b) List the elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_3$, and then list the subgroup $\langle (x, y) \rangle$ generated by each elements $(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_3$. What is the order of every element in $\mathbb{Z}_2 \times \mathbb{Z}_3$? Is $\mathbb{Z}_2 \times \mathbb{Z}_3$ cyclic?

   (c) We could analogously define $G_1 \times G_2 \times \cdots \times G_k$ for k groups, instead of 2.

      i. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

      ii. In general, if each group G_i has n_i elements, how many elements does the group $G = G_1 \times \cdots \times G_k$ have?

2. Let's prove some things:

   (a) If $G = G_1 \times G_2$, prove that G is abelian if and only if each factor is abelian.

   (b) If $G = G_1 \times G_2$, and $x \in G_1$ and $y \in G_2$ have finite order, prove that

   $$o(x, y) = \text{lcm}(o(x), o(y)),$$

   where lcm means *least common multiple.*

   Then, check that this theorem gives you the same answer for the orders of the elements in $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ in Problem 1.

(c) If $G = G_1 \times G_2$, and G_1 and G_2 are cyclic groups of finite order, prove that G is cyclic if and only if $|G_1|$ and $|G_2|$ are relatively prime.

(d) Generalize (a), (b), and (c) to direct products $G_1 \times G_2 \times \cdots \times G_k$.

3. Some applications of the theorems:

(a) Prove that $G = D_3 \times \mathbb{Z}_4$ is not abelian.

(b) Prove that $G = \mathbb{Z}_3 \times \mathbb{Z}_8$ is cyclic, and find an element $(x, y) \in G$ that is a generator.

(c) Find the order of $(2, 3, 4) \in \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_9$.

(d) Is $G = \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_9$ cyclic?

(e) Is Is $G = \mathbb{Z}_9 \times \mathbb{Z}_{17} \times \mathbb{Z}_{200}$ cyclic?

(f) Find an abelian group G with 12 elements where every element has order at most 6.

(g) Find a non-abelian group G with 12 elements where every element has order at most 6.

(h) Find an abelian group G with 24 elements where every element has order at most 6.

(i) Find a non-abelian group G with 24 elements where every element has order at most 6.