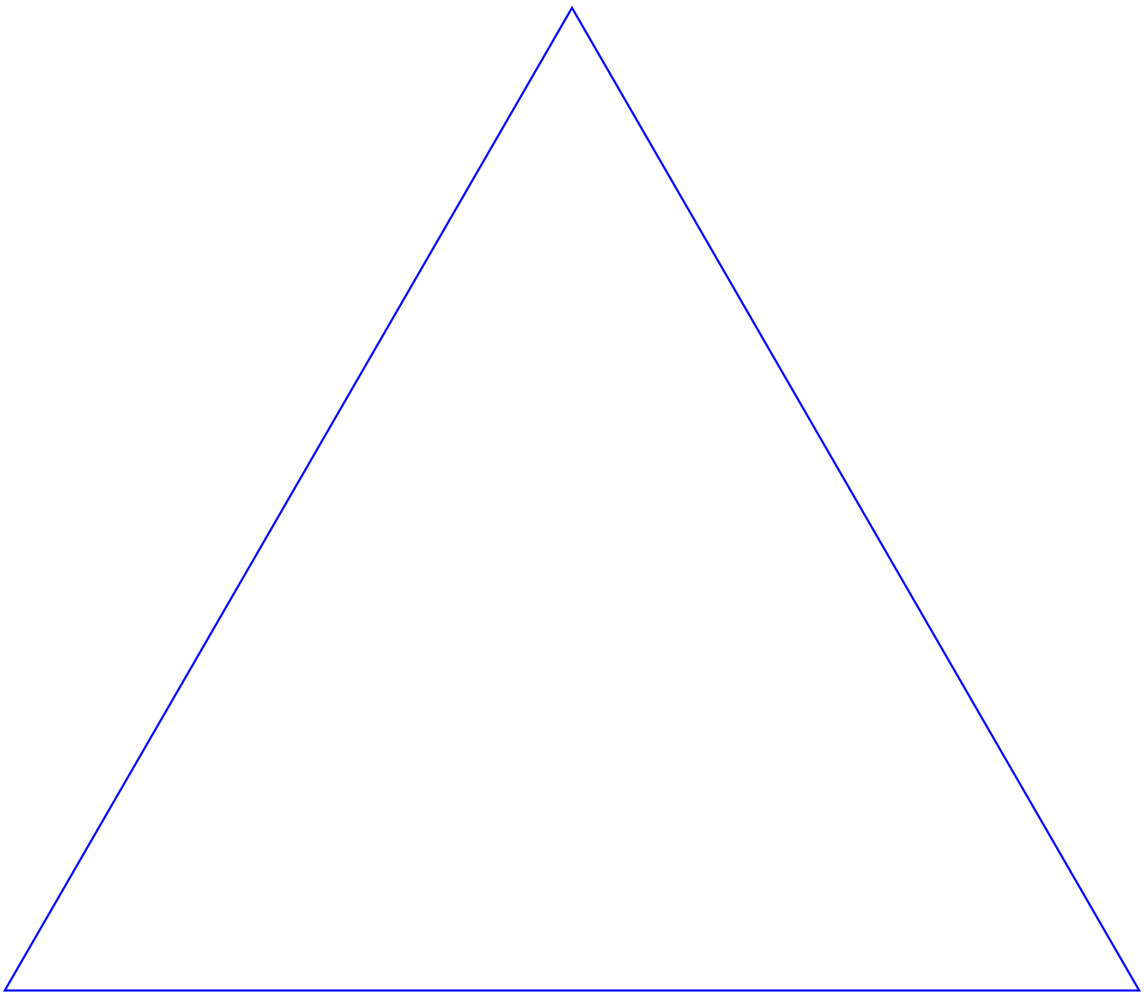
# Worksheet 2: Groups of Symmetries

We will explore **symmetries** of different shapes today. It will be helpful to have a model of one of the simplest shapes. Take a piece of paper and cut out/rip/fold an (approximately) equilateral triangle, large enough for you to move around easily. Don't want to make your own? Look, there's one right here! Feel free to rip along the edges or fold the paper.



**Definition 0.1.** A symmetry of a figure is a *rigid motion* that maps a figure to itself.

Imagine you have cut the triangle out of this piece of paper. A symmetry is an operation you can perform on the triangle so that it fits exactly back into the hole it was cut from.

1. How many symmetries does the equilateral triangle have? (Hint: use your triangle and perform rigid motions of it.) Come up with a description of each, including a label for each one.

   I will use the following list/description of the symmetries, but you can use whatever you prefer:

   - $e$ = do nothing operation
   - $r_1$ = rotation 120 degrees clockwise
   - $r_2$ = rotation 120 degrees counterclockwise
   - $s_1$ = flip across vertical axis of symmetry (axis connecting top vertex to bottom edge)
   - $s_2$ = flip across upper left to lower right axis of symmetry (axis connecting left edge to lower right vertex)
   - $s_3$ = flip across lower left to upper right axis of symmetry (axis connecting lower left vertex to right edge)

   Another way to list the operations is to draw the triangle and indicate where the vertices move to.

   Another way to list the elements is to write $r$ = rotation 120 degrees clockwise, and then $s$ = flip across vertical axis, and then write the symmetries as the composition of these two elements, like: $\{e, r, r \circ r, s, r \circ s, r \circ r \circ s\}$ or $\{e, r, r \circ r, s, s \circ r, s \circ r \circ r\}$.

2. Prove that you have found all symmetries of the triangle.

   A symmetry is uniquely determined by the configuration of the triangle at the end of applying the motion. There are 6 possible configurations of the triangle corresponding to the 6 possible labelings of the vertices (3 choices for the top vertex; 2 choices for the right vertex; and 1 choice left for the left vertex gives $3 \cdot 2 \cdot 1 = 6$ choices for triangle positioning). Because we have found six symmetries that yield these six relabelings, we have found all possible symmetries of the triangle.

3. Let $D_3$ denote the set of symmetries of the triangle and let $\circ$ denote composition, i.e. if $f$ and $g$ are two symmetries of the triangle, $f \circ g$ is the symmetry obtained by first doing the rigid motion corresponding to $g$ and then doing the rigid motion corresponding to $f$. You may assume that composition is an associative binary operation. Prove that $(D_3, \circ)$ is a group.

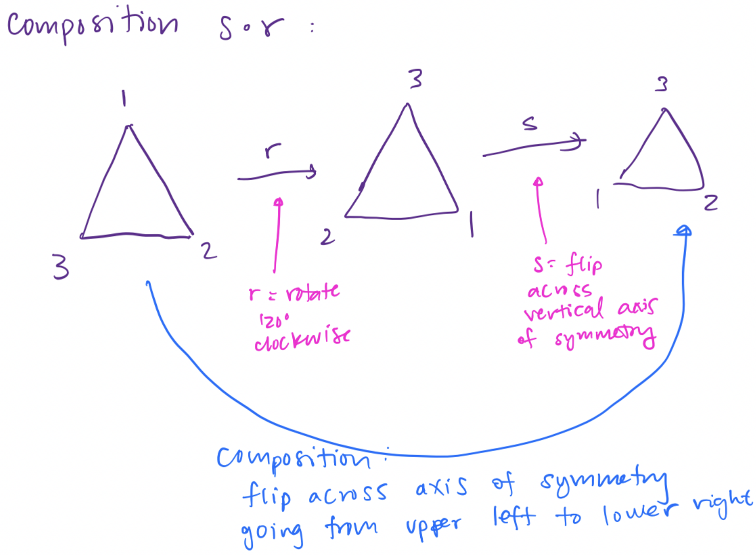
   By assumption, $\circ$ is an associative binary operation, so we just need to check the identity and inverse properties. By definition, $e$ is an operation that does nothing, so $e \circ f = f \circ e = f$ for every $f \in D_3$, so $e$ is the identity.

   To check the inverse property, we just list the inverse of each element to verify it exists. By definition, the inverse of $e$ is $e$. Because rotating left and then right (or right and then left) leaves the triangle in its original position, we have $r_1 \circ r_2 = r_2 \circ r_1 = e$, so the inverse of $r_1$ is $r_2$ and the inverse of $r_2$ is $r_1$. Because flipping the triangle over along one axis and then flipping it again along the same axis results in the original position, we have $s_i \circ s_i = e$ for each $i = 1, 2, 3$, so the inverse of any flip $s_i$ is itself. Therefore, inverses of each element exist, so $D_3$ is a group.

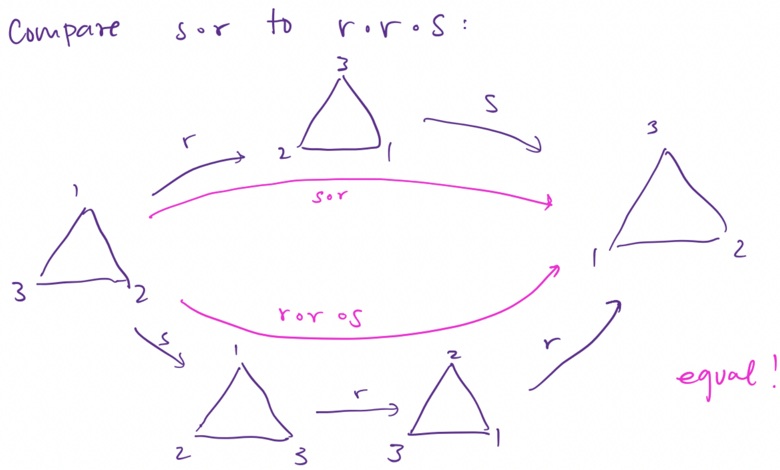
4. Let $s$ stand for flipping across the vertical axis and $r$ stand for rotation 120 deg clockwise.

(a) Show that every operation that you have already found can be written as a combination of (potentially multiple) $s$'s and $r$'s.

We do this by listing: identity $e = r \circ r \circ r$ or $e = s \circ s$; rotation clockwise is $r$; rotation counterclockwise is $r \circ r$; flip across vertical axis is $s$; flip across axis of symmetry going from upper left to lower right is $s \circ r$; flip across axis of symmetry going from lower left to upper right is $r \circ s$. You can verify all of these by actually moving the triangle! For example, here is a diagram for $s \circ r$:




(b) Show that $s \circ r = r \circ r \circ s$. (You may "show" this directly by moving your triangle.) Here is a diagram illustrating the equivalence:




5. Make a table for the binary operation $\circ$ on $D_3$:

- Leave the top left square blank.
- Along the top row, list all six symmetry operations using their symbol.
- Along the left column, list all six symmetry operations using their symbol (in the same order as the top row).
- In each empty square, fill it in with the symmetry operation you get from $f \circ g$, where $f$ is the operation in the left column and $g$ is the operation in the top row.
- After you fill in the table, list at least three observations about it.

| | $e$ | $r_1$ | $r_2$ | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $r_1$ | $r_2$ | $s_1$ | $s_2$ | $s_3$ |
| $r_1$ | $r_1$ | $r_2$ | $e$ | $s_3$ | $s_1$ | $s_2$ |
| $r_2$ | $r_2$ | $e$ | $r_1$ | $s_2$ | $s_3$ | $s_1$ |
| $s_1$ | $s_1$ | $s_2$ | $s_3$ | $e$ | $r_1$ | $r_2$ |
| $s_2$ | $s_2$ | $s_3$ | $s_1$ | $r_2$ | $e$ | $r_1$ |
| $s_3$ | $s_3$ | $s_1$ | $s_2$ | $r_1$ | $r_2$ | $e$ |

6. Using the table above, is $(D_3, \circ)$ an abelian group?

   No; we see from the table that the composition $s \circ r$ is flipping across the axis going from upper left to lower right, which is not the same as the composition $r \circ s$ which is flipping across the axis from the lower left to upper right. So, $s \circ r \neq r \circ s$.

7. Given any operation $f$, the **order** of $f$ is the number of times $f$ must be repeated to get the identity (do nothing) operation. Find the order of each symmetry operation in $D_3$. Do you notice anything?

   The order of $e$ is 1; the order of either rotation is 3, and the order of each flip is 2. One thing to notice: these are all factors of 6 (the size of the group!).

8. How many symmetry operations does a square have? List the operations and, if you're feeling inspired, make a table for the symmetries of the square like you did for the triangle. Call this set $D_4$. Prove that $(D_4, \circ)$ is a non-abelian group.

   A square has 8 operations. It is a group for the same reasoning as above, but non abelian because again $s \circ r$ is not equal to $r \circ s$ (where $s$ is the flip across any axis of symmetry and $r$ is rotation by 90 degrees).

9. If you label the vertices of the square $A, B, C, D$, can you get all re-labelings of the vertices of the square by performing symmetry operations? Prove your answer is correct.

   No! Because symmetries are rigid motions, and the vertices $A$ and $B$ are connected by an edge of the square, any symmetry will maintain the connection between $A$ and $B$. So we cannot get any relabeling where $A$ and $B$ are separated (like, $A, C, B, D$).

10. For $n \geq 3$, how many symmetry operations does a regular $n$-gon have? Prove your answer is correct, and prove that the set of symmetries of the $n$-gon $D_n$ is a non-abelian group.

    $D_n$ has $2n$ elements. One proof: a symmetry is determined by the ending configuration of the $n$-gon. There are $n$ choices for what vertex to put at the top of the figure, and once that vertex is fixed, we can either lay the shape down or flip it over (switching the two adjacent vertices), giving two choices for the final position. In other words, there are $n$ choices for how far to rotate the shape, and then the final configuration is determined by if we flip it over or not. Therefore, there are $n \cdot 2 = 2n$ configurations of the $n$-gon, so $2n$ elements of $D_n$.

11. Orient a regular $n$-gon so one vertex is at the top center of the figure. If $s$ is reflection across the vertical axis of symmetry of a regular $n$-gon and $r$ is rotation by $2\pi/n$ degrees clockwise, prove that $s \circ r = r^{n-1} \circ s$, where $r^{n-1}$ means the composition of $r$ $n-1$ times. (See: 4(b))

    This can be proven with a diagram like that in 4(b).

12. Using 9 and/or 10, list all elements of $D_n$ in form $r^i \circ s^j$ for some non-negative integers $i$ and $j$ (where anything to the zeroth power is defined to be the identity).

    The elements of $D_n$ can be written as $D_n = \{e, r, r^2, \ldots, r^{n-1}, s, s \circ r, s \circ r^2, \ldots, s \circ r^{n-1}\}$ or $D_n = \{e, r, r^2, \ldots, r^{n-1}, s, r \circ s, r^2 \circ s, \ldots, r^{n-1} \circ s\}$. (These are exactly the final positionings of the $n$-gon described in 10).

13. Bonus thought problems: think about symmetries of three-dimensional shapes. Start with a regular tetrahedron, and then move to a cube.

    We'll save this for later! Come talk to me if you're interested!