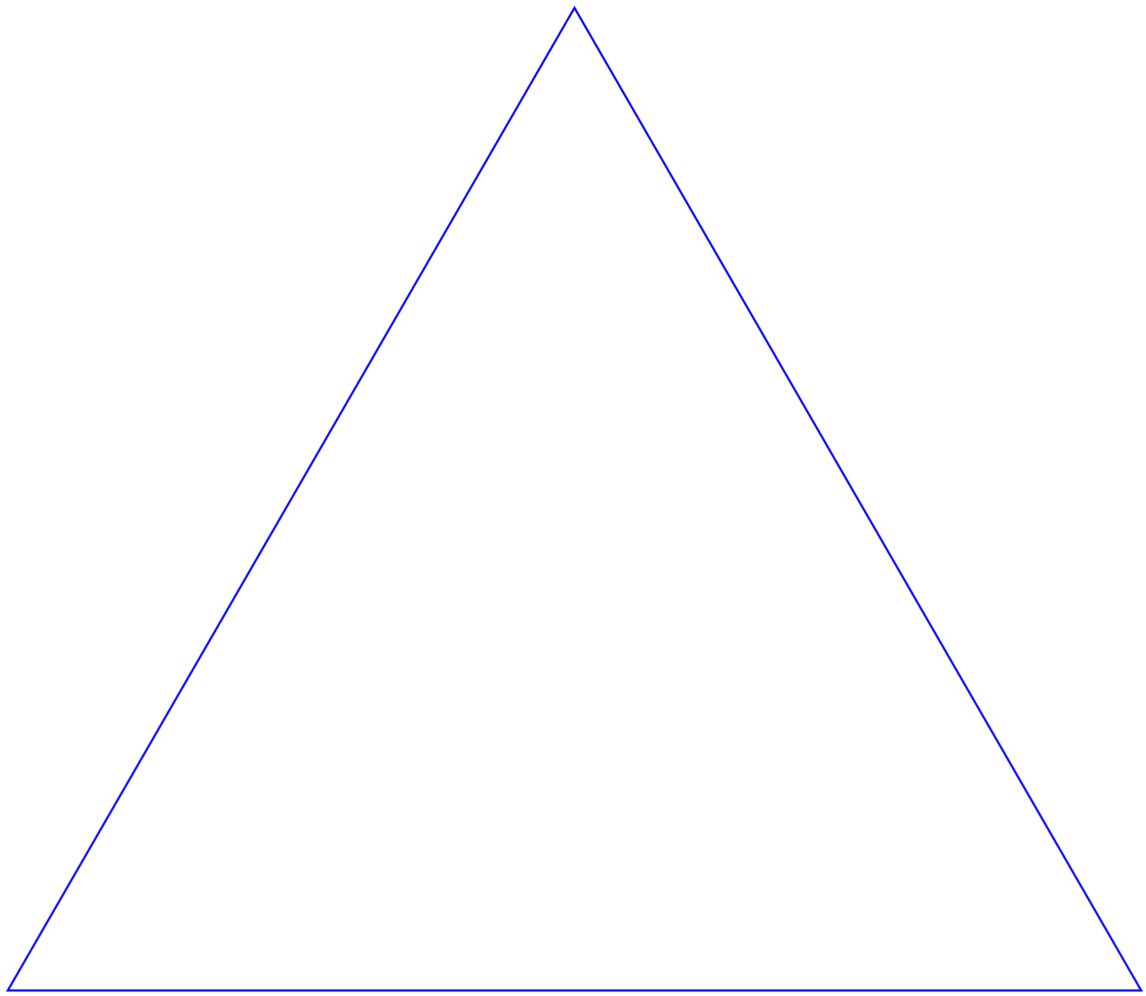
Worksheet 2: Groups of Symmetries

We will explore **symmetries** of different shapes today. It will be helpful to have a model of one of the simplest shapes. Take a piece of paper and cut out/rip/fold an (approximately) equilateral triangle, large enough for you to move around easily. Don't want to make your own? Look, there's one right here! Feel free to rip along the edges or fold the paper.



Definition 0.1. A symmetry of a figure is a *rigid motion* that maps a figure to itself.

Imagine you have cut the triangle out of this piece of paper. A symmetry is an operation you can perform on the triangle so that it fits exactly back into the hole it was cut from.

1. How many symmetries does the equilateral triangle have? (Hint: use your triangle and perform rigid motions of it.) Come up with a description of each, including a label for each one.

2. Prove that you have found all symmetries of the triangle.

3. Let D_3 denote the set of symmetries of the triangle and let $\circ$ denote composition, i.e. if f and g are two symmetries of the triangle, $f \circ g$ is the symmetry obtained by first doing the rigid motion corresponding to g and then doing the rigid motion corresponding to f. You may assume that composition is an associative binary operation. Prove that $(D_3, \circ)$ is a group.

4. Let s stand for flipping across the vertical axis and r stand for rotation 120 deg clockwise.

   (a) Show that every operation that you have already found can be written as a combination of (potentially multiple) s's and r's.

   (b) Show that $s \circ r = r \circ r \circ s$. (You may "show" this directly by moving your triangle.)

5. Make a table for the binary operation $\circ$ on D_3:

   - Leave the top left square blank.
   - Along the top row, list all six symmetry operations using their symbol.
   - Along the left column, list all six symmetry operations using their symbol (in the same order as the top row).
   - In each empty square, fill it in with the symmetry operation you get from $f \circ g$, where f is the operation in the left column and g is the operation in the top row.
   - After you fill in the table, list at least three observations about it.

6. Using the table above, is $(D_3, \circ)$ an abelian group?

7. Given any operation f, the **order** of f is the number of times f must be repeated to get the identity (do nothing) operation. Find the order of each symmetry operation in D_3. Do you notice anything?

8. How many symmetry operations does a square have? List the operations and, if you're feeling inspired, make a table for the symmetries of the square like you did for the triangle. Call this set D_4. Prove that $(D_4, \circ)$ is a non-abelian group.

9. If you label the vertices of the square A, B, C, D, can you get all re-labelings of the vertices of the square by performing symmetry operations? Prove your answer is correct.

10. For $n \geq 3$, how many symmetry operations does a regular n-gon have? Prove your answer is correct, and prove that the set of symmetries of the n-gon D_n is a non-abelian group.

11. Orient a regular n-gon so one vertex is at the top center of the figure. If s is reflection across the vertical axis of symmetry of a regular n-gon and r is rotation by $2\pi/n$ degrees clockwise, prove that $s \circ r = r^{n-1} \circ s$, where r^{n-1} means the composition of r $n-1$ times. (See: 4(b))

12. Using 9 and/or 10, list all elements of D_n in form $r^i \circ s^j$ for some non-negative integers i and j (where anything to the zeroth power is defined to be the identity).

13. Bonus thought problems: think about symmetries of three-dimensional shapes. Start with a regular tetrahedron, and then move to a cube.