Worksheet 1: Binary Operations

1. Which of the following are binary operations on the given set? (You should be able to explain why or why not these are binary operations.)

   (a) $\cdot$ (multiplication) on $S = \{1\}$
   This is a binary operation on S; $1 \cdot 1 = 1$ is always defined and is in S.

   (b) $\cdot$ on $S = \{1, -1\}$
   Homework!

   (c) $\cdot$ on $S = \{2^n \mid n \in \mathbb{Z}\}$
   This is a binary operation on S: if $s_1 = 2^n$ and $s_2 = 2^m$, then $s_1 \cdot s_2 = 2^n 2^m = 2^{n+m}$, which is in S.

   (d) $\star$ on $\mathbb{Z}$ given by $a \star b = a^2 + b + 1$
   This is a binary operation on $\mathbb{Z}$: for any $a, b \in \mathbb{Z}$, $a^2 + b + 1$ is always defined and is an integer.

   (e) $a * b = ab$ on $S = \{1, 6, 3, 2, 18\}$
   Homework!

   (f) $\cdot$ on $\mathbb{C}$ (to do this, you may want to figure out what the product of two complex numbers is, like $(a + bi)(c + di)$)
   This is a binary operation: given two complex numbers $a+bi$ and $c+di$, their product is
   $$(a + bi)(c + di) = ac + bci + adi - bd = (ac - bd) + (ad + bc)i.$$
   This is of the form (real number) + (real number)i, so is an element of $\mathbb{C}$.

   (g) $\cdot$ on $S = \{a + bi \in \mathbb{C} \mid a^2 + b^2 = 1\}$
   This is a binary operation on $\mathbb{C}$, so we know the product of any two complex numbers is again a complex number. To be a binary operation on S, we must check that the product of two elements in S is still in S. So, suppose $a+bi \in S$, meaning $a^2+b^2 = 1$, and $c + di \in S$, meaning $c^2 + d^2 = 1$. Then, by the previous part, we know
   $$(a + bi)(c + di) = ac + bci + adi - bd = (ac - bd) + (ad + bc)i.$$
   For this to be an element of S, we must have $(ac-bd)^2 + (ad+bc)^2 = 1$. Let us check if this is true:
   $$\begin{aligned}(ac - bd)^2 + (ad + bc)^2 &= a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2 \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= a^2(c^2 + d^2) + b^2(c^2 + d^2) \\ &= (a^2 + b^2)(c^2 + d^2) \\ &= 1 \cdot 1 \quad \text{because of our assumption on } a, b, c, d \\ &= 1.\end{aligned}$$
   Therefore, $(ac - bd) + (ad + bc)i \in S$, so multiplication is a binary operation on S.

2. For a set X, the *power set* of X is the set $\mathcal{P}(X)$ of all subsets of X.

   (a) If $X = \{1, 2\}$, find $\mathcal{P}(X)$.

   $$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

   (b) For $A, B \in \mathcal{P}(X)$, define $A \star B = A \cap B$ (the intersection of A and B). Is $\star$ a binary operation on $\mathcal{P}(X)$?

   $\star$ is a binary operation on $\mathcal{P}(X)$ because the intersection of two subsets of X is still a subset of X.

   (c) For $A, B \in \mathcal{P}(X)$, define $A \Delta B = (A - B) \cup (B - A)$. (Reminder/definition: $A - B$ means the elements in A that are not in B; $B - A$ means the elements in B that are not in A; and $\cup$ means union). Is Δ is a binary operation on $\mathcal{P}(X)$?

   Δ is also a binary operation: by definition, $A \Delta B$ consists of certain elements in A with certain elements of B, and because A and B are subsets of X, $A \Delta B$ is a subset of X. Therefore, $A \Delta B \in \mathcal{P}(X)$.

3. For each of the following binary operations, determine if it is associative and/or commutative.

   (a) $a \star b = a - b$ on $\mathbb{Z}$

   This is not commutative or associative. For general $a, b, c \in \mathbb{Z}$, $a - b \neq b - a$, and $(a - b) - c = a - b - c \neq a - (b - c) = a - b + c$.

   (b) $a \star b = a + b - ab$ on $\mathbb{Z}$

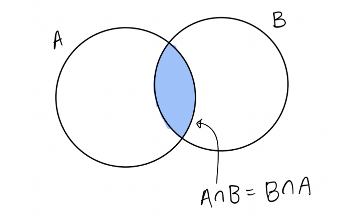
   Homework!

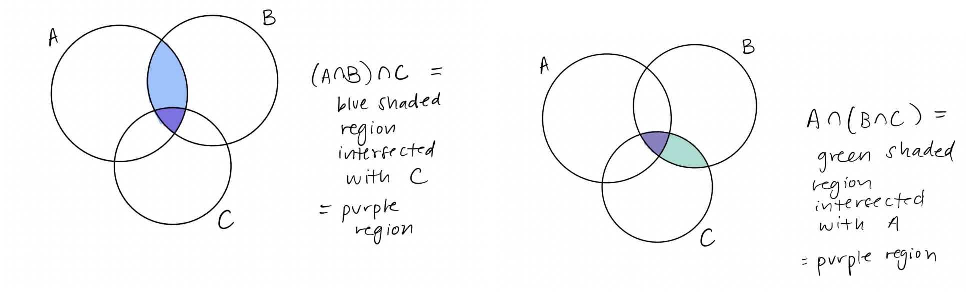
   (c) $a \star b = b$ on $\mathbb{Z}$

   Homework!

   (d) The operations in 2(b) and 2(c) on $\mathcal{P}(X)$ in problem 3

   For 2(b), $\cap$ is both commutative and associative: $A \cap B = B \cap A$ and $(A \cap B) \cap C = A \cap (B \cap C)$. Proof-by-picture:

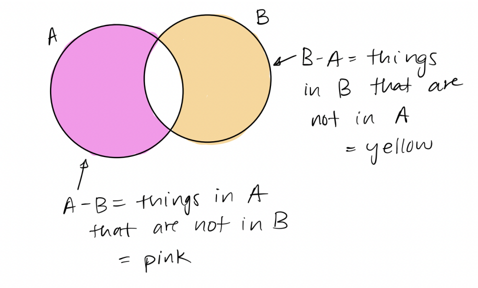




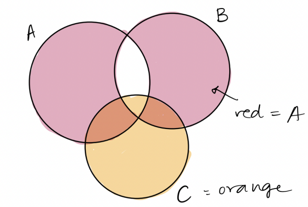
Similarly, Δ is both commutative and associative:

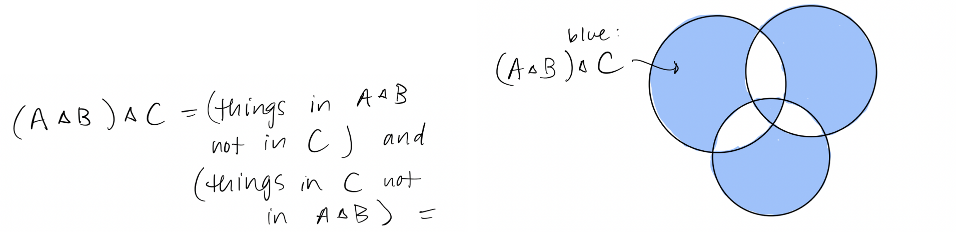
$$A \Delta B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \Delta A$$

(see picture, too)



and for associativity, draw the diagram:





As an exercise, verify that you get the same blue region for $A\Delta(B\Delta C)$ to show that $(A\Delta B)\Delta C = A\Delta(B\Delta C)$.

4. For *finite sets*, we can make a table describing the binary operation. For example, if $S = \{a, b, c\}$, we can list the elements in the top row and left-most column of the table, and then we fill in the table with the element obtained by applying the binary operation of the element in the left column with the element in the top row:

$\star$	a	b	c
a	$a \star a$	$a \star b$	$a \star c$
b	$b \star a$	$b \star b$	$b \star c$
c	$c \star a$	$c \star b$	$c \star c$

(a) Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Let $a \star b = a + b \pmod{n}$ (where $\pmod{n}$ means the remainder when dividing by n). For $n = 4$, fill in the table for the binary operation $a + b \pmod 4$ on $\mathbb{Z}_4$.

Homework!

(b) For $X = \{1, 2\}$, fill in the table for the binary operation $\cap$ on $\mathcal{P}(X)$. (Bonus: make a second table for Δ.)

$\cap$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{1\}$	$\emptyset$	$\{1\}$
$\{2\}$	$\emptyset$	$\emptyset$	$\{2\}$	$\{2\}$
$\{1,2\}$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$

Bonus table for Δ:

Δ	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\emptyset$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{1\}$	$\{1\}$	$\emptyset$	$\{1,2\}$	$\{2\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\emptyset$	$\{1\}$
$\{1,2\}$	$\{1,2\}$	$\{2\}$	$\{1\}$	$\emptyset$

(c) From looking at the table for a general binary operation $\star$ on a set S, how would you determine if $\star$ is commutative?

The operation $\star$ is commutative if and only if $s_1 \star s_2 = s_2 \star s_1$. These are the same element if and only if the table is symmetric across the main diagonal (going from upper left to lower right).