Worksheet 1: Binary Operations

1. Which of the following are binary operations on the given set? (You should be able to explain why or why not these are binary operations.)

   (a) $\cdot$ (multiplication) on $S = \{1\}$

   (b) $\cdot$ on $S = \{1, -1\}$

   (c) $\cdot$ on $S = \{2^n \mid n \in \mathbb{Z}\}$

   (d) $\star$ on $\mathbb{Z}$ given by $a \star b = a^2 + b + 1$

   (e) $a * b = ab$ on $S = \{1, 6, 3, 2, 18\}$

   (f) $\cdot$ on $\mathbb{C}$ (to do this, you may want to figure out what the product of two complex numbers is, like $(a + bi)(c + di)$)

   (g) $\cdot$ on $S = \{a + bi \in \mathbb{C} \mid a^2 + b^2 = 1\}$

2. For a set X, the *power set* of X is the set $\mathcal{P}(X)$ of all subsets of X.

   (a) If $X = \{1, 2\}$, find $\mathcal{P}(X)$.

   (b) For $A, B \in \mathcal{P}(X)$, define $A \star B = A \cap B$ (the intersection of A and B). Is $\star$ a binary operation on $\mathcal{P}(X)$?

   (c) For $A, B \in \mathcal{P}(X)$, define $A \Delta B = (A - B) \cup (B - A)$. (Reminder/definition: $A - B$ means the elements in A that are not in B; $B - A$ means the elements in B that are not in A; and $\cup$ means union). Is Δ is a binary operation on $\mathcal{P}(X)$?

3. For each of the following binary operations, determine if it is associative and/or commutative.

   (a) $a \star b = a - b$ on $\mathbb{Z}$

   (b) $a \star b = a + b - ab$ on $\mathbb{Z}$

   (c) $a \star b = b$ on $\mathbb{Z}$

   (d) The operations in 2(b) and 2(c) on $\mathcal{P}(X)$ in problem 2

4. For *finite sets*, we can make a table describing the binary operation. For example, if $S = \{a, b, c\}$, we can list the elements in the top row and left-most column of the table, and then we fill in the table with the element obtained by applying the binary operation of the element in the left column with the element in the top row:

$\star$	a	b	c
a	$a \star a$	$a \star b$	$a \star c$
b	$b \star a$	$b \star b$	$b \star c$
c	$c \star a$	$c \star b$	$c \star c$

   (a) Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Let $a \star b = a + b \pmod{n}$ (where $\pmod{n}$ means the remainder when dividing by n). For $n = 4$, fill in the table for the binary operation $a + b \pmod 4$ on $\mathbb{Z}_4$.

   (b) For $X = \{1, 2\}$, fill in the table for the binary operation $\cap$ on $\mathcal{P}(X)$. (Bonus: make a second table for Δ.)

   (c) From looking at the table for a general binary operation $\star$ on a set S, how would you determine if $\star$ is commutative?