Math 411 Practice Problems for Exam 1

Reminder: Exam 1 is on Tuesday, March 5, in class.

Here are several practice problems for the first exam. There are **more problems here than will be on the actual exam. The actual exam will be 4 - 5 questions.** Solutions will be posted later this week. *All homework problems and worksheet problems should also be considered practice problems for the exam.*

1. (Practice with definitions.)
   (a) Define a group.
   (b) Define an abelian group.
   (c) Define a subgroup of a group G.

2. Which of the following are groups? If your answer is yes, prove it is a group, and if your answer is no, justify why not.
   (a) $G = \mathbb{Z}$ with binary operation $a \star b = ab - 2$
   (b) $G = \mathbb{R} - \{0\}$ with binary operation $a \star b = ab/(a^2 + b^2)$
   (c) $G = \mathcal{P}(X)$ (the power set of a set X), $A \star B = A \cap B$.

3. For each of the following questions, determine if it is true or false. If it is true, provide a brief reason, and if it is false, provide a counterexample.
   (a) Suppose G is a group and $a, b \in G$. If $a^{-1}b = e$, then $a = b$.
   (b) If G is a finite group with n elements, then there is an element x in G such that $o(x) = n$.
   (c) A subgroup of an abelian group is always abelian.
   (d) If every proper subgroup of a group G is abelian, then G is abelian.

4. (a) Find the order and inverse of the element $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ in $(M_2(\mathbb{R}), +)$.
   (b) Find the order and inverse of the element $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ in $(GL_2(\mathbb{R}), \cdot)$
   (c) Find the order and inverse of the element r (rotation by 120 degrees clockwise) in the group D_3 that was symmetries of the equilateral triangle.
   (d) Find the order and inverse of the element 6 in the group $\mathbb{Z}_9$.
   (e) Find $\langle x \rangle$ in the group $(\mathbb{Q}^{>0}, \times)$ where $x = \frac{1}{2}$.
   (f) Find $\langle x \rangle$ in the group $\mathbb{Z}_{10}$ where $x = 3$.

5. For each of the following questions, give an example of a group or subgroup with the desired properties and briefly explain or show why your answer satisfies the given properties.
   (a) A group with four elements with more than one subgroup with two elements.
   (b) An abelian group that is not cyclic.
   (c) An infinite non-abelian group.

6. Is the set of 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $a, c, d \in \mathbb{R}$, $ad \neq 0$, a subgroup of $GL_2(\mathbb{R})$? If your answer is yes, prove that it is correct. If not, explain why not.

7. Let G be a finite group and let $x \in G$. Prove that the set $H = \{x^n \mid n \in \mathbb{Z}, n > 0\}$ is a subgroup of G. Give a counterexample to show that this is not true if G is not finite.

8. If H is a subgroup of $\mathbb{Z}$ that contains two elements $a, b \in H$ such that $\gcd(a, b) = 1$, prove that $H = \mathbb{Z}$.

9. Let G be a group and $x, y \in G$. Assume that $x \neq e$, $o(y) = 2$, and $yxy^{-1} = x^2$. Find $o(x)$ and prove your answer is correct.

10. Let G be a group and $g \in G$. Let $Z(g) = \{x \in G \mid gx = xg\}$. Is $Z(g)$ a subgroup of G?

11. Let $G = (\mathbb{Q}, +)$. Prove that G is not cyclic.