Math 411 Homework 4

Due March 17, 2024 to Gradescope (by midnight)

1. Problem 5.4: For each group below, how many subgroups does it have? What are they?
   (a) $\mathbb{Z}_{18}$
   (b) $\mathbb{Z}_{35}$
   (c) $\mathbb{Z}_{36}$
2. Problem 5.5: Find all of the subgroups of Q (the quaternion group). Show that Q is an example of a nonabelian group with the property that all its proper subgroups are cyclic.
3. Problem 5.7: Let $G = \langle x \rangle$ be cyclic of order n. Show that x^m is a generator of G if and only if $\gcd(m, n) = 1$.
4. Problem 5.19: Let $G = \langle x \rangle$ be an infinite cyclic group. Prove that all of the distinct subgroups of G are $\langle e \rangle$, $\langle x \rangle$, $\langle x^2 \rangle$, $\langle x^3 \rangle$, ... In other words, prove that every subgroup of G is one of these, and no two of them are the same.
5. Problem 5.20: Let G be a finite group with no subgroups other than $\{e\}$ or G itself. Prove that either $G = \{e\}$ or G is a cyclic group of prime order. (Hint: if $G \neq \{e\}$, then there is some $x \in G$ such that $\langle x \rangle \subset G \ldots$)
6. Problem 6.1: Calculate the order of the element.
   (a) $(4, 9) \in \mathbb{Z}_{18} \times \mathbb{Z}_{18}$
   (b) $(7, 5) \in \mathbb{Z}_{12} \times \mathbb{Z}_8$
   (c) $(8, 6, 4) \in \mathbb{Z}_{18} \times \mathbb{Z}_9 \times \mathbb{Z}_8$
   (d) $(8, 6, 4) \in \mathbb{Z}_9 \times \mathbb{Z}_{17} \times \mathbb{Z}_{10}$
7. Problem 6.2: Which of the following groups are cyclic?
   (a) $\mathbb{Z}_{12} \times \mathbb{Z}_9$
   (b) $\mathbb{Z}_{10} \times \mathbb{Z}_{85}$
   (c) $\mathbb{Z}_4 \times \mathbb{Z}_{25} \times \mathbb{Z}_6$
   (d) $\mathbb{Z}_{22} \times \mathbb{Z}_{21} \times \mathbb{Z}_{65}$
8. Problem 6.6: If $G = G_1 \times G_2 \times \cdots \times G_n$, prove that G is abelian if and only if each factor is abelian.
9. Problem 6.7: Construct a nonabelian group of order 16, and one of order 24.
10. Problem 6.8: Construct a group of order 81 with the property that every element except the identity has order 3.