Math 411 Homework 2

Due February 18, 2024 to Gradescope (by midnight)

The problem numbers below refer to Saracino's text, second edition. This homework covers Section 2 and Section 3, although some problems are not from the book. You can neatly handwrite or type your homework, and do not need to copy the problem statement, but please clearly label each problem that you are working on. If you collaborate with others, please write their names at the top of your assignment.

1. Problem 3.4: Let g be an element of a group $(G, \star)$ such that for some element $x \in G$, $x \star g = x$. Show that $g = e$.
2. Problem 3.6: Prove the cancellation laws (Theorem 3.6), which state:

   Let $(G, \star)$ be a group and let $x, y, z \in G$. Then,

   (a) if $x \star y = x \star z$, then $y = z$, and

   (b) if $y \star x = z \star x$, then $y = z$.
3. Problem 3.7: Let $(G, \star)$ be a finite group and consider the table for the binary operation of G. Show that every element of G occurs precisely once in each row and column of the table.
4. Problem 3.9: Show that G is abelian if and only if $(x * y)^{-1} = x^{-1} * y^{-1}$ for every $x, y \in G$.
5. Problem 3.11: Let G be a group such that $x^2 = e$ for all $x \in G$. Show that G is abelian.
6. Determine all groups with two elements, as follows. Suppose $G = \{e, a\}$ is a group with two elements, where e is the identity. Using the first two steps below, determine the only possible binary operation on G by writing the table, and answer one additional question:

   (a) Using the identity property, fill in all possible squares that are $e \star \_\_$ or $\_\_ \star e$.

   (b) Using previous results (inverse property, uniqueness of identity/inverses, problems 3, 4, 5, ...) fill in the final remaining square. Justify your answer.

   (c) Are all groups with two elements abelian?
7. Using the outline in the previous problem, show that there is only one group with three elements $G = \{e, a, b\}$ by writing down the only possible table for the binary operation. Justify your answer. Are all groups with three elements abelian?
8. Last but not least, we will consider groups of order four. We do this in steps:

   (a) If $(G, \star)$ is a finite group and has an even number of elements, prove that is some element $x \in G$, $x \neq e$, such that $x \star x = e$ (i.e. $x^{-1} = x$). (Possible hint: try to pairs up elements in G with their inverses.)

   (b) By (a), if G has 4 elements $G = \{e, a, b, c\}$, one must be its own inverse. Let a be this element, i.e. suppose $a \star a = e$. With this information, show that there are exactly two possible binary operations on G by writing down the corresponding tables. Justify your answer. Are all groups with four elements abelian?