FEBRUARY 15 NOTES

Last time, we proved many things about groups. Today, we will consider several examples of groups.

First, some notation: it gets cumbersome to write $\star$ all the time; e.g. $x \star y$ can be tedious to write over and over again. So, when dealing with *abstract* groups, we will denote the operation by $x \star y = xy$. We will also write powers of a given element by x^n, where:

$$x^1 = x$$
$$x^2 = x \star x$$
$$x^3 = x \star (x \star x)$$
$$\dots$$

and so on.

So, whenever you see the notation xy in this class, it always means $x \star y$, where $\star$ is whatever the binary operation is. When we know the operation, we may choose to change notation. For example, if the operation is $+$, we will write $x + y$ instead of xy. For powers, instead of writing x^2, which means $x \star x$, we would write $2x$ because $x + x = 2x$.

Now, to define a few more groups:

The Quaternions.

The first group we will define is the quaternions. Let

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

and define the binary operation to be *multiplication*, which we assume to be associative, where:

- For any $x \in Q$, $1x = x1 = x$, and
- for any $x \in Q$, $(-1)x = x(-1) = -x$, and
- for any $x, y \in Q$, $(-x)y = x(-y) = -(xy)$, and
- $i^2 = j^2 = k^2 = -1$, and
- $ij = k$, $jk = i$, and $ki = j$.

From here, we can determine every other possible product. For instance, starting with $jk = i$, we could multiply both sides by j to get $j^2k = ji$, and then use that $j^2 = -1$ to get $ji = -k$.

We'll start out today by filling in the multiplication table for the quaternions:

*	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	1	-1
-k	-k	k	-j	j	i	-i	-1	1

Some questions to answer from the table? What is the inverse of every element in Q? Is the operation commutative? We'll discuss Q much more in the future!

For the rest of the day, we will introduce a very important group in the form of a worksheet. (If you're reading the notes, please see Worksheet 2!)