K-MODULI OF FANO VARIETIES AND LOG FANO PAIRS

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These notes and exercises grew out of lectures delivered at the 2022 AGNES Summer School on Higher Dimensional Moduli and then the 2023 FRG Special Month at the University of Michigan. My intention is that students with a first course in algebraic geometry, at the level of Hartshorne, will be able to learn the foundations of K-moduli theory by working through the text and exercises at the end of each section. Many of the exercises are hands-on and will work through explicit examples of stable or unstable objects in K-moduli spaces.

This is ultimately intended to be part of a larger, more comprehensive manuscript with Dori Bejleri on moduli of higher dimensional varieties in general.

Contents

For what follows, we will be working over $\mathbb{C}$, although it is an intriguing question to consider K-stability in characteristic p.

1. Introduction to K-stability

Our first goal is to introduce the notion of K-stability with connections to other invariants and classify K-(semi)stable smooth del Pezzo surfaces.

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1.1. **Fano varieties and history of K-stability.**

Definition 1.1. A smooth variety X is called a **Fano** variety if $-K_X$ is ample (i.e. for some $m \gg 0$, the rational map $|-mK_X| : X \dashrightarrow \mathbb{P}(H^0(-mK_X))$ is an embedding).

If $\dim X = 1$, X is Fano if and only if $X = \mathbb{P}^1$. In general, $\mathbb{P}^n$ is Fano for any n, but there are many other types of Fano varieties in higher dimensions. When $\dim X = 2$, a Fano surface is called a *del Pezzo* surface.

Before we define K-stability, we detour into its historical origins.

It is an old(er) question in differential geometry to study when Fano varieties can be equipped with a Kähler-Einstein (KE) metric. We won't be using this perspective in this series of lectures, but it provides relevant background on how K-stability came to be.

A smooth Kähler variety with Kähler form ω is said to have a KE metric if ω satisfies the Einstein equation

$$\mathrm{Ric}(\omega) = \lambda\omega$$

for some constant λ. In the typical trichotomy of varieities–(K_X ample, trivial, or antiample–a smooth projective variety with ample canonical class always admits a KE metric, proved independently by Aubin and Yau in 1978 [Aub78, Yau78], and one with trivial canonical class always admits a KE metric, proved by Yau [Yau78]. However, for Fano varieties, it was known much earlier that they cannot always admit a KE metric. For example, in 1957, Matsushima proved that if X is KE, then $\mathrm{Aut}(X)$ is reductive [Mat57]. Therefore, it was of interest to differential geometers to formulate a notion for Fano varieties that precisely captured the existence of a KE metric.

Several years later, the notion of K-stability was introduced. In 1992, Ding and Tian [DT92] introduced the generalized Futaki invariant to capture the existence of a KE metric, and proved that the existence of such a metric implies this invariant is non-negative. In 1997, Tian [Tia97] (analytically) and later Donaldson [Don02] in 2002 (algebraically), the notion of K-stability was formally defined using the Futaki invariant, and the *Yau-Tian-Donaldson Conjecture* was made: a smooth Fano variety is K-polystable if and only if it admits a KE metric. This conjecture was proven by Chen, Donaldson, and Sun in 2012 [CDS15a, CDS15b, CDS15c], and has since been extended beyond the smooth case. We will define K-stability below, but you may be wondering:

Question 1.2. What does this have to do with algebraic geometry?

As we'll see shortly, the algebraic formulation of K-stability looks like other powerful notions in algebraic geometry (for example, GIT), so +1 for motivation to study it. Also, it has something to do with degenerating varieties in families, so +1 for connecting to moduli problems. However, it is *remarkable* that this differential geometric notion is exactly the correct thing to study to get well-behaved moduli spaces of Fano varieties, and *remarkable* that it has so many connections to older algebro-geometric concepts (e.g. singularities and the minimal model program).

In the words of Chenyang Xu, *"The concept of K-stability is one of the most precious gifts differential geometers brought to algebraic geometers."*

1.2. **K-stability via test configurations.** Without further ado, let's define K-stability. We will not restrict ourselves to the smooth world; let us consider arbitrary normal projective varieties.

Definition 1.3 (Tian, Donaldson)**.** Let (X, L) be a polarized projective variety of dimension n, and suppose X is normal. Because L is ample, for $m \gg 0$, there is an embedding $|L^m| : X \to \mathbb{P}^N$. For any action $\mathbb{G}_m$ on PGL_{N+1}, there in an induced action $\mathbb{G}_m$ on the class $[X] \in \mathrm{Hilb}(\mathbb{P}^N)$. Let $[X_0] = \lim_{t\to 0} t \cdot [X]$.

A **test configuration** is the induced family

$$\begin{tikzcd}
\mathbb{G}_m \times (X, L^m) \arrow[r] \arrow[d] & (\mathcal{X}, \mathcal{L}) \arrow[d] \\
\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} \arrow[r] & \mathbb{A}^1
\end{tikzcd}$$

Given a test configuration, by Riemann-Roch, we can compute

$$\begin{aligned}
d_k := h^0(X, L^k) &= a_0 k^n + a_1 k^{n-1} + \dots \\
&= \frac{L^n}{n!} k^n - \frac{L^{n-1} \cdot K_X}{2(n-1)!} k^{n-1} + \dots .
\end{aligned}$$

Since $\mathbb{G}_m$ is acting on $(\mathcal{X}, \mathcal{L})$, is it acting on $(\mathcal{X}_0, \mathcal{L}_0) =: (X_0, L_0)$, and hence $H^0(X_0, L_0)$. For $k \gg 0$, the total weight of this action on $H^0(X_0, L_0^{k/m})$ also grows as a polynomial:

$$w_k = b_0 k^{n+1} + b_1 k^n + \dots .$$

Remark 1.4. To compute the b_i and show that this is a polynomial, we can complete the family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{A}^1$ to a family $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ over $\mathbb{P}^1$ by adding the trivial fiber (X, L^m) over $\infty \in \mathbb{P}^1$. We do this by gluing the family $(\mathcal{X}, \mathcal{L})$ to the trivial family $X \times \mathbb{P}^1 \setminus \infty$ along $\mathbb{A}^1 \setminus 0$. Then, we have a test configuration $(\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1$ with a $\mathbb{G}_m$ action and can use equivariant Riemann-Roch to compute the weight:

$$\begin{aligned}
w_k &= b_0 k^{n+1} + b_1 k^n + \dots \\
&= \frac{\overline{\mathcal{L}}^{(n+1)/m}}{(n+1)!} k^{n+1} - \frac{\overline{\mathcal{L}}^{(n/m)} \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}}{2n!} k^n + \dots .
\end{aligned}$$

For related discussion, see [Xu23, §2.1.2].

Definition 1.5 (Tian, Donaldson)**.** The **generalized Futaki invariant** $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$ of the test configuration $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0^2}.$$

This expression comes from the quotient

$$F(k) := \frac{w_k}{k d_k} = F_0 + F_1 \frac{1}{k} + F_2 \frac{1}{k^2} + \dots$$

and

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = -F_1.$$

While the invariant may look complicated, it is very closely related to Hilbert stability and Chow stability (which are defined in similar ways, by Mumford).

In practice, for X Fano, we use $L = -mK_X$. We will exclusively use this in what follows, and $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$ takes on a particularly nice form [Xu23, Prop 2.17]:

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{1}{2(-K_X)^n} \left(\left(\frac{1}{m} \overline{\mathcal{L}} \right)^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \frac{n}{n+1} \left(\frac{1}{m} \overline{\mathcal{L}} \right)^{n+1} \right).$$

We will use the Futaki invariant to define K-stability.

Definition 1.6 (Tian, Donaldson)**.** Let X be a variety such that $-K_X$ is ample. X is

(1) K-semistable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$.
(2) K-stable if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$, and equality holds if and only if $(\mathcal{X}, \mathcal{L})$ is trivial (outside of a codimension 2 locus).
(3) K-polystable if X is K-semistable and, if $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = 0$, then $\mathcal{X} \cong X \times \mathbb{A}^1$ (outside of a codimension 2 locus).

Remark 1.7. By [BX19], if X is K-stable, then $\mathrm{Aut}(X)$ is finite, and if $\mathrm{Aut}(X)$ is finite, then X is K-stable if and only if it is K-polystable. We will explore K-polystability in more detail in later sections. In light of the first sentence, if you are familiar with GIT, think of 'polystable' as a closed orbit condition as it is in GIT.

If the complexity of the definition did not scare you off, hopefully here you are seeing a RED FLAG. This definition depends on the $\mathbb{G}_m$ action and the power m used for L!

Remark 1.8 (Red Flag!)**.** To test if a variety is K-(semi/poly)stable, we must a priori test *infinitely* many test configurations, which depend on the $\mathbb{G}_m$ action *and* the power m used in the embedding $|L^m| : X \to \mathbb{P}^N$. (For those familiar with GIT: this is like checking the Hilbert-Mumford weight for every possible embedding of X into a higher and higher projective space.) How can this possibly be reasonable?

We can begin to simplify this making connections to other quantities in algebraic geometry. First, although nothing about singularities explicitly appears in the test configuration definition, asking that a variety is K-semistable has (surprising!) consequences on the singularities of X.

Definition 1.9. A projective variety X is $\mathbb{Q}$-Fano if X has log terminal singularities (which implies that some multiple of X is $\mathbb{Q}$-Cartier) and $-K_X$ is ample.

Theorem 1.10 ([Oda13])**.** *If X is normal and $-K_X$ is ample, then K-semistability of X implies that X has log terminal singularities. In other words, if X is K-semistable, it is $\mathbb{Q}$-Fano.*

We can also restrict the test configuration definition to sufficiently"nice" varieties X_0!

Definition 1.11. A test configuration $(\mathcal{X}, \mathcal{L})$ is called a **special test configuration** if $\mathcal{X}$ is a $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties, i.e. $\mathcal{L} \sim -mK_{\mathcal{X}}$ and X_0 has klt singularities.

Theorem 1.12 ([LX14])**.** *To test K-(semi/poly)stability, one only needs to test special test configurations.*

We give a very brief idea of the proof; for more, see [LX14].

Proof. Main idea: use the MMP! Starting with any test configuration $(\mathcal{X}, \mathcal{L})$, we can perform birational modifications like MMP operations or finite base change and normalization to produce a special test configuration $(\mathcal{X}^s, -mK_{\mathcal{X}^s})$. Then, show that the Futaki invariant can only decrease under these birational operations. □

Remark 1.13. If $(\mathcal{X}, \mathcal{L})$ is a special test configuration,

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = -\frac{1}{2(-K_X)^n(n+1)} \left(-K_{\overline{\mathcal{X}}/\mathbb{P}^1}\right)^{n+1}.$$

So, for special test configurations, the sign of the Futaki invariant is determined by the sign of $\left(-K_{\overline{\mathcal{X}}/\mathbb{P}^1}\right)^{n+1}$.

We can therefore restrict to 'nice' families in the definition of K-stability, but there are still infinitely many test configurations to check! We need several other invariants to better study K-stability of varieties.

1.3. **K-stability via the α-invariant.** In the rest of this section, we will primarily focus on *other invariants* that capture K-(semi)stability. These can be easier to check in practice.

We first define the α-invariant, introduced by Tian in [Tia87]. The original definition is analytic, but by Theorem A.3 of Demailly's appendix in [CS08], it coincides with what follows.

Definition 1.14 (Tian)**.** Let X be a $\mathbb{Q}$-Fano variety. Tian's α-invariant is

$$\alpha(X) = \inf_{0 \leq D \sim_{\mathbb{Q}} -K_X} \mathrm{lct}(X, D).$$

Example 1.15. If $X = \mathbb{P}^n$, because $-K_{\mathbb{P}^n} = (n+1)H$,

$$\alpha(\mathbb{P}^n) = \frac{1}{n+1}.$$

Theorem 1.16 ([Tia87])**.** *Let X be a $\mathbb{Q}$-Fano variety of dimension n. If*

$$\alpha(X) > (\geq) \frac{n}{n+1}$$

then X is K-(semi) stable.

This is not an if-and-only-if, but can be readily computable, and we can use the α invariant to check K-stability of Fano varieties.

Example 1.17. If $X = \mathbb{P}^1$, $\alpha(X) = \frac{1}{2}$ so $\mathbb{P}^1$ is K-semistable. But, for $n > 1$, the α-invariant tells us nothing. We will see later that a refinement of this criterion can be used to show that $\mathbb{P}^n$ is always K-semistable.

Let's use this to understand the stability of some del Pezzo surfaces.

Definition 1.18. The **degree** of a del Pezzo surface X is $d = (-K_X)^2$. For $X = \mathbb{P}^2$, $d = 9$. For $X = \mathbb{P}^1 \times \mathbb{P}^1$, $d = 8$.

Example 1.19. Let X be a del Pezzo surface of degree 1 (so X is the blow up of $\mathbb{P}^2$ at 8 points). We will directly compute $\alpha(X)$ and show that X is K-stable.

Consider the linear system $|-K_X|$. In terms of curves on $\mathbb{P}^2$,

$$-K_X = \pi^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{8} E_i$$

so the curves in $|-K_X|$ are (strict transforms of) cubic curves in $\mathbb{P}^2$ that pass through all 8 points that were blown up. Suppose that $D \sim_{\mathbb{Q}} -K_X$. If $\mathrm{Supp} D \notin |-K_X|$, pick $x \in D$ and choose a curve $C \in |-K_X|$ such that $x \in C$. Because $D \sim_{\mathbb{Q}} -K_X$ and $C \sim_{\mathbb{Q}} -K_X$, $D \cdot C = (-K_X)^2 = 1$, so the multiplicity of D at every point is at most 1. This implies that (X, D) is log canonical, so $\mathrm{lct}(X, D) = 1$.

Then, to finish computing $\alpha(X)$, we just need to compute the log canonical threshold of curves $D \in |-K_X|$. Each such D is a cubic plane plane curve vanishing at the 8 points we blew up (so, in particular, D must be reduced and irreducible, because no three of the points blown up were co-linear, and no 6 were on a conic), and such curves are either:

smooth

nodal

cuspidal

and the log canonical threshold of a pair (X, D) with D as above is:

$$\begin{aligned}
&\text{smooth} : \operatorname{lct}(X, D) = 1 \\
&\text{nodal} : \operatorname{lct}(X, D) = 1 \\
&\text{cuspidal} : \operatorname{lct}(X, D) = \frac{5}{6}.
\end{aligned}$$

Therefore, in each case, we see that $\alpha(X) = \min \operatorname{lct}(X, D) \geq \frac{5}{6} > \frac{2}{3}$ so by Theorem 1.16, X is K-stable.

Remark 1.20. More generally, Cheltsov [Che08] has shown that $\alpha(X) \geq \frac{2}{3}$ for X a del Pezzo surface of degree ≤ 4, hence any such X is K-semistable.

There are many refinements of Tian's criteria. None of these are if-and-only-ifs, but at least they make it possible to check K-stability in many cases.

Theorem 1.21 ([Fuj19a])**.** *If X is a surface, or smooth of dimension ≥ 3, and*

$$\alpha(X) \geq \frac{n}{n+1},$$

then X is K-stable.

Corollary 1.22. All del Pezzo surfaces of degree ≤ 4 are K-stable.

Example 1.23. If X is a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1$, then Cheltsov and Park [CP02] have shown $\alpha(X) \geq \frac{n}{n+1}$. Therefore, by Theorem 1.21, all such X are K-stable.

What about $\mathbb{P}^n$? Surely we should be able to determine if it is semistable or not. We use a refinement of the α-invariant criterion to help us out–a G-invariant version.

Definition 1.24. Let X be a Fano variety with a group action by an algebraic group G. Define

$$\alpha_G(X) = \inf_{0 \leq D \sim_{\mathbb{Q}} -K_X, D \text{ is } G\text{-invariant}} \operatorname{lct}(X, D).$$

Proved in increasing levels of generality by [DS16, LX20, LZ22, Zhu21], we have the following.

Theorem 1.25. *Let X be a Fano variety with a group action by an algebraic group G.*

(1) If

$$\alpha_G(X) \geq \frac{n}{n+1},$$

then X is K-semistable.

(2) If G is reductive and

$$\alpha_G(X) > \frac{n}{n+1},$$

then X is K-polystable.

Example 1.26. Let $X = \mathbb{P}^n$ and $G = \mathrm{PGL}(n+1)$. There are no G-invariant divisors, hence $\alpha_G(X) = \infty$, so $\mathbb{P}^n$ is K-polystable.

1.4. **K-stability via the β and δ invariants.** Thus far, we have introduced the test configuration definition for K-stability and the α-invariant (and α_G) which we could use as a test to determine K-stability. The α-invariant had the advantage that it is relatively computable, however, it has a distinct disadvantage of not being an if-and-only-if statement. Can we get such a statement? Because Theorem 1.10 ties K-semistability to the singularities and birational geometry of X, we may hope for a definition of K-(semi/poly)stability in more birational geometric terms. In fact, we can connect the Futaki invariant to Fujita and Li's β-invariant (or δ invariant) . For the reader's convenience, note that in [Xu23], the β-invariant is called the Fujita-Li invariant.

Definition 1.27. Let X be a $\mathbb{Q}$-Fano variety and E a prime divisor over X. Let $\mu : Y \to X$ be any morphism such that $E \subset Y$.

Let $A_X(E)$ be the log discrepancy of the divisor E, or the number

$$A_X(E) = 1 + \mathrm{ord}_E(K_Z - f^*K_X) = 1 + a_X(E).$$

Define $S_X(E)$ to be

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tE)dt.$$

This does not depend on choice of μ and Y, so we often write

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(-K_X - tE)dt.$$

The β-invariant of the divisor E is

$$\beta_X(E) = A_X(E) - S_X(E).$$

The δ-invariant of E is

$$\delta_X E = \frac{A_X(E)}{S_X(E)}.$$

In the integral S, we must compute $\mathrm{vol}(D)$ for $D = -K_X - tE$. What follows are some notions related to volumes of divisors.

Definition 1.28. The volume a divisor D on a normal variety X of dimension n is

$$\mathrm{vol}(D) = \lim_{m\to\infty} \frac{h^0(X, mD)}{m^n/n!}.$$

Definition 1.29. A divisor D on a normal variety X of dimension n is **big** if one of the following equivalent definitions hold:

(1) For $m \gg 0$, the map given by the linear system $|mD| : X \dashrightarrow \mathbb{P}^N$ is birational onto its image.
(2) For $m \gg 0$, there exists a constant $c > 0$ such that $h^0(X, mD) > cm^n$.
(3) $\mathrm{vol}(D) > 0$.

How do we compute volumes? If D is a divisor on a normal variety X of dimension n, we have the following:

(1) If D is nef, $\mathrm{vol}(D) = D^n$.
(2) If D is big, by definition $\mathrm{vol}(D) > 0$, and we can at least bound the volume of D from below by considering the image of the linear system $|mD| : X \dashrightarrow \mathbb{P}^N$. Because D is big, the image is a variety Y birational to X. If we assume this a morphism $f : X \to Y$, some divisors in X may be contracted. If we write $D = f^*\mathcal{O}(1) + N$ for some effective divisor N supported on the contracted locus, then $\mathrm{vol}(f^*(\mathcal{O}(1)) \leq \mathrm{vol}(D)$ (because $h^0(f^*\mathcal{O}(1)) \subset h^0(D)$). And, $f^*\mathcal{O}(1)$ is nef, so its volume is just $\mathcal{O}(1)^n$. Therefore, we compute a lower bound for the

volume of D by the volume of $\mathcal{O}(1)$. In practice, we determine N by considering MMP-like birational modifications of X. If the divisor D is trivial or negative on some effective curve in $\overline{NE(X)}$, then we contract the class of this curve. If this is divisorial, N will be supported on the divisor, and if it is a small contraction, we flip or flop the class of the curve to a new variety $X \dashrightarrow X^+$ and pullback everything to a common partial resolution $\widehat{X}$. The divisor N will be supported on the exceptional divisors of $\widehat{X} \to X$.

For context, for surfaces, this is called a *Zariski decomposition*: we can always write a big divisor D on a surface X as $D = P + N$, where P is nef, N is negative (meaning it has negative definite intersection matrix/is contractible, or is 0), and $P \cdot N = 0$. In this case, we have the equality $\mathrm{vol}(D) = \mathrm{vol}(P) = P^2$. To relate this to the birational modifications above, because N is negative, it corresponds to a contractible curve in the surface, and P is the resulting divisor on the contraction.

We will see some examples of computing volumes in the exercises.

Remark 1.30. In the definition of $S_X(E)$, we need to compute an improper integral. However, $\mathrm{vol}(-\mu^* K_X - tE) > 0$ if and only if $-\mu^* K_X - tE$ is big. In terms of divisors on Y, the closure of the big cone of divisors is the pseudo-effective cone, so the volume is only non-zero if $t \in [0, \tau]$ where τ is the pseudo-effective threshold. This is finite; at some point we have subtracted 'too much' E and the divisor is no longer pseudo-effective. Therefore, we could re-write

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\tau \mathrm{vol}(-K_X - tE)dt.$$

With this definition, we can relate the K-(semi/poly)stability of X *intrinsically* to the birational geometry of X! The following theorem is usually called the **valuative criterion** for K-(semi/poly)stability. Initially given by Fujita and Li, there are several important contributions necessary connecting uniform K-stability and K-stability by the other cited authors.

Theorem 1.31 ([Fuj19b, Li17, FO18, BJ20, LXZ22])**.** *A variety X is K-semistable (resp. stable) if and only if $\beta_X(E) \geq 0$ (resp. > 0) for all prime divisors E over X (equivalently, $\delta_X(E) \geq 1$ (resp. > 1)).*

This should be taken with a (possibly smaller) RED FLAG than the test configuration definition, because while it may appear more birational geometric in nature, it still requires checking every divisor E over X! However, it is a nice complement to the α-invariant criteria because it allows us to prove that certain varieties are *not* (semi)stable. Such examples can be found in the exercises. We provide a sample computation below.

Example 1.32. Let's compute $\beta_{\mathbb{P}^2}(E)$ where E is the exceptional divisor of a blow up of a point on $\mathbb{P}^2$. Let $\mu : Y \to \mathbb{P}^2$ be the blow up. Because $K_Y = \mu^*(K_{\mathbb{P}^2}) + E$, we have $A_{\mathbb{P}^2}(E) = 1 + 1 = 2$.

Now we need to compute $S_{\mathbb{P}^2}(E)$. We know $(-K_{\mathbb{P}^2})^2 = 9$. To compute $\mathrm{vol}(-K_{\mathbb{P}^2} - tE)$, we need to understand when the divisor is ample, big, and nef. The volume is non-zero when the divisor is big (by definition of being big!).

We know that $\mu^*(-K_{\mathbb{P}^2}) - tE = -K_Y + (1-t)E$. The Mori cone of Y is generated by E and the class of a fiber F of the ruled surface $Y \to \mathbb{P}^1$. Because $(-K_Y + (1-t)E) \cdot E = 1 - (1-t) = t$, this is positive on E for $t > 0$. Similarly, $(-K_Y + (1-t)E) \cdot F = 2 + (1-t) = 3 - t$, so this is positive on F for $t < 3$. Because this is ample exactly when it has positive intersection with both F and E, this is ample for $0 < t < 3$. Also, when $t = 3$, this is trivial on F, and the morphism induced by the linear system $|m(-K_Y + (1-t)E)|$ therefore contracts F and hence contracts Y to a curve. Because this is not birational for any $m > 0$, the divisor is not big for any $t \geq 3$.

This implies that:

$$\text{for } 0 \leq t \leq 3, \quad \operatorname{vol}(\mu^*(-K_{\mathbb{P}^2}) - tE) = (\mu^*(-K_{\mathbb{P}^2}) - tE)^2 = 9 - t^2$$
$$\text{for } t \geq 3, \quad \operatorname{vol}(\mu^*(-K_{\mathbb{P}^2}) - tE) = 0.$$

So, we can compute $S_{\mathbb{P}^2}(E)$:

$$S_{\mathbb{P}^2}(E) = \frac{1}{9}\int_0^3 (9 - t^2)dt = \frac{18}{9} = 2.$$

Finally, we can conclude that $\beta(E) = A(E) - S(E) = 0$.

Note that this alone does not imply that $\mathbb{P}^2$ is K-semistable; to use the valuative criterion, we need to show that $\beta(E) \geq 0$ for *every* divisor E over $\mathbb{P}^2$.

Remark 1.33. The tools in this section are sufficient to characterize the stability of del Pezzo surfaces.

We list the stability of each del Pezzo surface, along with a reason.

degree	stability	reason
9	polystable	equivariant α-invariant
8 ($X = \mathbb{P}^1 \times \mathbb{P}^1$)	polystable	equivariant α-invariant, Exercise 5
8 ($X = \mathbb{F}_1$)	unstable	β-invariant computation, Exercise 7
7	unstable	β-invariant computation, Exercise 7
6	polystable	equivariant α-invariant, Exercise 5
5	stable	equivariant α-invariant plus finite $\operatorname{Aut}(X)$, [Che08]
≤ 4	stable	α-invariant, Cor. 1.22

1.5. Exercises.

(1) (a) If X is the blow up of $\mathbb{P}^2$ at k sufficiently general points, $0 \leq k \leq 8$, or $X = \mathbb{P}^1 \times \mathbb{P}^1$, prove that X is Fano.
(b) Determine what 'sufficiently general' means in the previous exercise (it should be a condition on the points that were blown up).
(c) Prove that every smooth Fano surface is one of those listed in (a). Such surfaces are called *del Pezzo* surfaces.

(2) Give an example of a smooth Fano variety whose automorphism group is non-reductive. By Matsushima's result [Mat57], this cannot be K-polystable.

(3) If X is the blow up of $\mathbb{P}^2$ at r general points, $0 \leq r \leq 8$, prove that the degree of X is $9 - r$.

(4) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow up of $\mathbb{P}^2$ at three points are K-polystable (hint: $\operatorname{Aut}(X)$-invariant divisors?).

(5) Show that the blow up of $\mathbb{P}^2$ at a point is K-unstable by computing $\beta_X(E)$, where $E \subset X$ is the exceptional divisor of the blow up.

(6) Show that the blow up of $\mathbb{P}^2$ at a point (a del Pezzo surface of degree 8) is K-unstable by computing $\beta_X(E)$, where $E \subset X$ is the exceptional divisor of the blow up.

(7) Show that the del Pezzo surface of degree 7 is K-unstable.

(8) By blowing up the cone point and computing $\beta_X(E)$, show that $\mathbb{P}(1, 1, n)$ is K-unstable for any $n > 1$.

(9) Explain why the blow up of $\mathbb{P}^3$ along a planar cubic is Fano, and show that it is K-unstable. (Hint: there are two natural divisors to check β of; the exceptional of the blow-up and the

strict transform of the plane containing the curve. Only one will work to show you it is K-unstable.)

(10) This exercise will introduce some ideas related to K-stability of pairs. Let $X = \mathbb{P}(1,1,2)$.
  (a) Show that the sections of $\mathcal{O}_X(2)$ define an embedding $X \hookrightarrow \mathbb{P}^3$ whose image is $(xy = z^2)$. This shows directly that $\mathbb{P}(1,1,2)$ is the singular quadric cone.
  (b) Show that X is K-unstable.
  (c) Given a log Fano pair (X, D) of dimension n, we can define K-stability of the pair. For any prime divisor E over X, define $\beta_{(X,D)}(E) = A_{(X,D)}(E) - S_{(X,D)}(E)$, where $A_{(X,D)}(E)$ is the log discrepancy of the pair, and for any morphism $\mu : Y \to X$ extracting E,
$$S_{(X,D)}(E) = \frac{1}{(-K_X - D)^n} \int_0^\infty \mathrm{vol}(\mu^*(-K_X - D) - tE) dt.$$
  Then, we say (X, D) is K-semistable if $\beta_{(X,D)}(E) \geq 0$ for every E.
  Let $c \in \mathbb{Q}^{>0}$. Let $D = cQ$, where Q is the hyperplane section at infinity of the cone $\mathbb{P}(1,1,2)$. Compute $\beta_{(X,cQ)}(E)$, where E is the exceptional divisor of the resolution, and compute $\beta_{(X,cQ)}(Q)$. Show that (X, cQ) is K-unstable for all $c \neq 1/2$.

2. Abban-Zhuang theory of admissible flags

Remark 2.1. *A note to the new reader: in a first effort to learn K-stability, I encourage you to skip this section as the notation is quite technical. The machinery is very powerful and in reality we are truly just computing dimensions of spaces of global sections, so please come back to this after building a baseline comfort level with K-stability and K-moduli.*

While the invariants β and δ are very useful for determining if Fano varieties are K-unstable, it is still very difficult to prove something is actually K-stable because you must check an inequality for every divisor E over your variety. One may ask how feasible it is to actually determine the stability of an arbitrary Fano variety. In general, this is incredibly difficult, but one method for checking K-stability that has proved to be extraordinarily useful is the theory of admissible flags introduced by Abban and Zhuang in [AZ22]. Very roughly, you may think about this as an adjunction result for K-stability: it allows you to restrict to smaller dimensional subvarieties and check appropriate inequalities there. This leads to an inductive approach to determine K-stability.

To use this theory, we first need to introduce the language of filtrations. We will do this using the language of log Fano pairs, but you may also set $\Delta = 0$ in what follows. For more details and justification of the following results, see [AZ22].

Definition 2.2. If L is a big line bundle on a variety X, the *graded linear series* $\mathbf{V}_\bullet = \{V_m\}_{m\in\mathbb{N}}$ to be $V_m = H^0(X, mL)$ for $m \in \mathbb{N}$ is called the **complete linear series associated to** L.

The **volume** of $\mathbf{V}_\bullet$ is $\mathrm{vol}(\mathbf{V}_\bullet) = \lim_{m\to\infty} \dim V_m/(m^n/n!) = \mathrm{vol}(L)$.

We will use $L = -K_X - \Delta$ in what follows. Now, we refine this series by a divisor E over X.

Definition 2.3. For any divisor E over X and positive real number t, define the linear series
$$(\mathcal{F}_E V_m)_t = \{s \in V_m \mid \mathrm{ord}_E(s) \geq mt\},$$
where we pullback $s \in V_m$ to a variety Y extracting E.

For a real number t,
$$\mathrm{vol}(\mathcal{F}_E \mathbf{V}_\bullet)_t = \lim_{m\to\infty} \dim(\mathcal{F}_E V_m)_t/(m^n/n!).$$
Then, let
$$S(\mathbf{V}_\bullet; E) = \frac{1}{\mathrm{vol}\mathbf{V}_\bullet} \int_0^\infty \mathrm{vol}(\mathcal{F}_E \mathbf{V}_\bullet)_t dt.$$

If (X,Δ) is a log Fano pair of $\dim X = n$ and $L = -K_X - \Delta$, perhaps convince yourself that this is just the definition of $S_{(X,\Delta)}(E)$.

Definition 2.4. As in the previous section, we define the δ-invariant

$$\delta(X,\Delta;L) := \delta(X,\Delta;\mathbf{V}_\bullet) = \inf_E \frac{A_{(X,\Delta)}(E)}{S(\mathbf{V}_\bullet;E)}.$$

Definition 2.5. Let Z be a subvariety of X. We define

$$\delta_Z(X,\Delta;\mathbf{V}_\bullet) = \inf_{E:Z\subset C_X(E)} \frac{A_{(X,\Delta)}(E)}{S(\mathbf{V}_\bullet;E)}.$$

It is clear that $\delta(X,\Delta;\mathbf{V}_\bullet) = \inf_{Z\subset X}\delta_Z(X,\Delta;\mathbf{V}_\bullet)$. Furthermore, (X,Δ) is K-semistable if $\delta_p(X,\Delta;\mathbf{V}_\bullet) \geq 1$ for all points $p \in X$.

Next, we 'restrict' to E:

Definition 2.6. Define the muligraded linear series

$$W^E_{m,j} = Im(H^0(Y, mL - jE)) \to H^0(E, mL|_E - jE|_E))$$

where $L|_E = -K_E - \Delta_E$ and Δ_E is the different, and $E|_E$ is a sensible divisor as long as E is 'nice' (precisely, we need E to be of plt type).

Then, define

$$\mathrm{vol}(\mathbf{W}^E_{\bullet\bullet}) = \lim_{m\to\infty}\sum_{j\geq 0}\dim W^E_{m,j}/(m^n/n!).$$

It is a theorem that in this set-up, $\mathrm{vol}(\mathbf{W}^E_{\bullet\bullet}) = \mathrm{vol}(\mathbf{V}_\bullet)$.

Now, if we refine this multigraded linear series by a divisor F over E, we analogously define the volume as in Definition 2.3.

Definition 2.7. For a divisor F over E and a positive number t, define

$$(\mathcal{F}_F W_{m,j})_t = \{s \in W^E_{m,j} \mid \mathrm{ord}_D(s) \geq mt\}$$

and define

$$\mathrm{vol}(\mathcal{F}_F\mathbf{W}^E_{\bullet\bullet})_t = \lim_{m\to\infty}\sum_{j\geq 0}\dim(\mathcal{F}_F W^E_{m,j})_t/(m^n/n!).$$

Finally, define

$$S(\mathbf{W}^E_{\bullet\bullet};F) = \frac{1}{\mathrm{vol}\mathbf{W}^E_{\bullet\bullet}}\int_0^\infty \mathrm{vol}(\mathcal{F}_F\mathbf{W}^E_{\bullet\bullet})_t dt.$$

Finally, we get to the adjunction-like result.

Theorem 2.8. *For a primitive divisor E over X and any $Z \subset X$ such that $Z \subset C_X(E)$, let $\pi : Y \to X$ be a prime blow-up extracting E. Then,*

$$\delta_Z(X,\Delta;\mathbf{V}_\bullet) \geq \min\left\{\frac{A_{(X,\Delta)}(E)}{S(\mathbf{V}_\bullet;E)}, \inf_{Z'}\delta_{Z'}(E,\Delta_E;\mathbf{W}^E_{\bullet\bullet})\right\}$$

where the second infimum is taken over all $Z' \subset Y$ such that $\pi(Z') = Z$, and

$$\delta_{Z'}(E,\Delta_E;\mathbf{W}^E_{\bullet\bullet}) = \inf_F \frac{A_{(E,\Delta_E)}(F)}{S(\mathbf{W}^E_{\bullet\bullet};F)}$$

where F is a prime divisor over E with $Z' \subset C_E(F)$.

This says that the δ-invariant of X can be bounded below by just the δ-value computed by E and then the δ-invariant of a smaller dimensional variety (E). Furthermore, it can be applied repeatedly to get an inductive result, reducing ultimately to the case Z' is a point in a curve E.

Remark 2.9. This method has been incredibly useful in the quest to determine K-(poly/semi) stability of every smooth Fano threefold. For example, this is readily employeed in [ACC+23] to determine the K-(poly/semi)stability of the general member of every deformation type of smooth Fano threefolds. In many cases, it is used to further show that *every* member (not just general ones) are K-(poly/semi)stable.

We will use this to show several varieties are K-semistable, but first will introduce a more formulaic version due to [ACC+23].

Suppose (X, Δ) is a klt pair with Δ effective. Suppose E is a prime divisor over X of *plt type*, i.e. there exists a morphism $\pi : \widetilde{X} \to X$ extracting E such that $-E$ is π-ample and $(\widetilde{X}, \Delta + E)$ is plt, where $\widetilde{\Delta}$ is the divisor satisfying

$$K_{\widetilde{X}} + \widetilde{\Delta} + (1 - A_{X,\Delta}(E))(E) = \pi^*(K_X + \Delta).$$

Define Δ_Y by

$$K_E + \Delta_E = (K_{\widetilde{X}} + \widetilde{\Delta} + E)|_E.$$

Then, as above,

$$\delta_Z(X, \Delta; \mathbf{V}_\bullet) \geq \min\left\{\frac{A_{(X,\Delta)}(E)}{S(\mathbf{V}_\bullet; E)}, \inf_{Z'} \delta_{Z'}(E, \Delta_E; \mathbf{W}^E_{\bullet\bullet})\right\}$$

where Z' runs over subvarieties of E. We can compute the first term: it is just $A_{X,\Delta}(E)/S_{X,\Delta}(E)$.

Furthermore, if X is a surface, then E is a curve, and must be smooth by the plt-type assumption. Therefore, Z' just ranges over points $p \in E$ with $\pi(p) = Z$, and if $p \in C_E(F)$, then $p = F$. So, we simply need

$$\delta_p(E, \Delta_E; \mathbf{W}^E_{\bullet\bullet}) = \frac{A_{E,\Delta_E}(p)}{S(\mathbf{W}^E_{\bullet\bullet}; p)}.$$

The numerator is just $1 - \mathrm{ord}_p(\Delta_E)$, and the denominator is computed as follows. Let τ be the pseudoeffective threshold of $\pi^*(-K_X - \Delta) - uE$ (the maximal u such that this is pseudoeffective), and let $P(u) = P(\pi^*(-K_X - \Delta) - uE)$ by the positive part of the Zariski decomposition of $\pi^*(-K_X - \Delta) - uE$ and $N(u) = N(\pi^*(-K_X - \Delta) - uE)$ the negative part. You would have already had to find these to compute $S_{X,\Delta}(E)$! Provided that E is not contained in the support of $N(u)$, then

$$S(\mathbf{W}^E_{\bullet\bullet}; p) = \frac{2}{\mathrm{vol} L} \int_0^\tau \int_0^\infty \mathrm{vol}(P(u)|_E - vp)\, dv du.$$

If $t(u)$ is the pseudoeffective threshold of $P(u)|_E - vp$, this is just

$$S(\mathbf{W}^E_{\bullet\bullet}; p) = \frac{2}{\mathrm{vol} L} \int_0^\tau \int_0^{t(u)} \max\{(\mathrm{ord}_p(P(u)|_E) - v), 0\}\, dv du.$$

Example 2.10. Let us use this theory to show every smooth cubic surface is K-semistable. Let $p \in X$ be a point in X. We will bound $\delta_p(X)$. Let $E \subset X$ be an anticanonical divisor $E \in |-K_X|$ through p (this exists as $-K_X$ is very ample–see the exercises).

Because $\Delta = 0$ and E is a curve on X, $A_X(E) = 1$. We can also compute $S_X(E)$:

$$
\begin{aligned}
S_X(E) &= \frac{1}{\mathrm{vol}(-K_X)} \int_0^\infty \mathrm{vol}(-K_X - tE)dt \\
&= \frac{1}{3} \int_0^1 (-K_X + tK_X)^2 dt \\
&= \frac{1}{3} \int_0^1 (1-t)^2(-K_X)^2 dt \\
&= \frac{1}{3} \int_0^1 3(1-t)^2 dt \\
&= \frac{1}{3}
\end{aligned}
$$

Therefore, $A_X(E)/S_X(E) = 3 > 1$.

Now, as $-K_X - uE$ is nef if and only if it is pseudoeffective if and only if $0 < u < 1$, $P(u) = -K_X - uE$ and $N(u) = 0$. Restricting to E, $P(u)|_E = (1-u)E|_E = 3(1-u)p$. Then, $\mathrm{ord}_p(P(u)|_E) = 3(1-u)$, so

$$
\begin{aligned}
S(\mathbf{W}^E_{\bullet\bullet}; p) &= \frac{2}{\mathrm{vol} L} \int_0^\tau \int_0^{t(u)} \max\{(\mathrm{ord}_p(P(u)|_E) - v), 0\} dv du \\
&= \frac{2}{3} \int_0^1 \int_0^{3(1-u)} (3(1-u) - v) dv du \\
&= 1.
\end{aligned}
$$

Therefore,

$$
\delta_p(E, \Delta_E; \mathbf{W}^E_{\bullet\bullet}) = \frac{A_{E,\Delta_E}(p)}{S(\mathbf{W}^E_{\bullet\bullet}; p)} = 1.
$$

So, for any $p \in X$,

$$
\delta_p(X; -K_X) = \delta_p(X; \mathbf{V}_\bullet) \geq \min\left\{ \frac{A_{(X,\Delta)}(E)}{S(\mathbf{V}_\bullet; E)}, \delta_p(E, \Delta_E; \mathbf{W}^E_{\bullet\bullet}) \right\} = \min\{3, 1\} = 1.
$$

Because $\delta(X; -K_X) = \inf_{p \in X} \delta_p(X; -K_X)$, we have proven $\delta(X; -K_X) \geq 1$ so X is K-semistable. In fact, one can prove these are actually K-stable using the last sentence of [AZ22, Thm. 1.2].

Example 2.11. Next, we use this to show that the pair $(\mathbb{P}(1,1,2), \frac{1}{2}Q)$ from Exercise 10 is K-semistable. For p a smooth point, take E to be a ruling through the point p. In this case, Z' will just equal p as in the previous example. For p the singular point, take E to be the exceptional divisor of the blow-up. In this case, the Z' will have to range through points on the curve E.

Suppose first that p is a smooth point of $X = \mathbb{P}(1,1,2)$ and let $E \subset X$ be a ruling through p. Because E is not contained in $\Delta = \frac{1}{2}Q$, $A_{X,\Delta}(E) = 1$. We can also compute $S_{X,\Delta}(E)$:

$$
\begin{aligned}
S_{X,\Delta}(E) &= \frac{1}{\mathrm{vol}(-K_X-\Delta)}\int_0^\infty \mathrm{vol}(-K_X-\Delta-tE)dt \\
&= \frac{2}{9}\int_0^3 (3-t)^2E^2dt \\
&= \frac{1}{9}\int_0^3 (3-t)^2dt \\
&= 1
\end{aligned}
$$

Therefore, $A_X(E)/S_X(E) = 1$.

Now, as $-K_X - \Delta - uE$ is nef if and only if it is pseudoeffective if and only if $0 < u < 3$, $P(u) = -K_X - \Delta - uE$ and $N(u) = 0$. Restricting to E, $P(u)|_E = (3-u)E|_E = \frac{(3-u)}{2}p$. Then, $\mathrm{ord}_p(P(u)|_E) = \frac{3-u}{2}$, so

$$
\begin{aligned}
S(\mathbf{W}^E_{\bullet\bullet};p) &= \frac{2}{\mathrm{vol}L}\int_0^\tau\int_0^{t(u)} \max\{(\mathrm{ord}_p(P(u)|_E)-v),0\}dvdu \\
&= \frac{4}{9}\int_0^3\int_0^{(3-u)/2}(\frac{(3-u)}{2}-v)dvdu \\
&= \frac{1}{2}.
\end{aligned}
$$

If $p \in \mathrm{Supp}\Delta_E$, then $A_{E,\Delta_E}(p) = \frac{1}{2}$, and otherwise $= 1$. Hence,

$$
\delta_p(E,\Delta_E;\mathbf{W}^E_{\bullet\bullet}) = \frac{A_{E,\Delta_E}(p)}{S(\mathbf{W}^E_{\bullet\bullet};p)} \geq 1.
$$

For any p other than the singular point, this proves that $\delta_p(X,\Delta;-K_X-\Delta) \geq 1$. To complete the proof, it suffices to show this inequality for the singular point $p \in X$. This is left to the exercises.

2.1. **Exercises.**

(1) Finish Example 2.11.

3. Results on moduli of K-semistable Fano varieties

Now that we understand how to show Fano varieties are (or are not) K-semistable, we will connect the ideas of K-stability with moduli of Fano varieties. We will discuss issues that arise in construction of moduli spaces of all Fano varieties, learn how K-stability provides a *good* moduli space, called a K-moduli spaces, and continue to develop tools to understand K-stability to identify members of K-moduli spaces.

Now that we have introduced K-stability, we will enumerate several results that make it a good notion for moduli. First, a discussion of moduli of Fano varieties in general and some examples of things we 'want' from a moduli space.

Definition 3.1. A family of varieties $\mathcal{X} \to T$ is **$\mathbb{Q}$-Gorenstein** if $K_{\mathcal{X}/T}$ is $\mathbb{Q}$-Cartier.

This is a condition we usually impose on moduli problems because it makes things nicely behaved (and we are typically 'allowed' to assume it from the MMP!). In fact, for technical reasons, we typically assume the *Kollár condition* that every reflexive power of $K_{\mathcal{X}/T}$ commutes with base change.

Example 3.2. In a $\mathbb{Q}$-Gorenstein family, $K_{\mathcal{X}/T}|_{\mathcal{X}_t} = K_{\mathcal{X}_t}$ is $\mathbb{Q}$-Cartier, so ampleness of $-K_{\mathcal{X}_t}$ is an open condition. We usually want conditions in our moduli problems to be open (or, at least locally closed) so that if something is satisfied for one fiber, it is also so for nearby fibers. This is essential when constructing moduli spaces.

Example 3.3. We generally want our moduli spaces to be proper; i.e. "limits exist in our moduli problem." In moduli of varieties of general type when K_X is ample, we saw in the first two chapters that we can use the minimal model program to do this. In practice, we do this by taking Proj of some canonical section ring $R(K_X)$, which is finitely generated by [BCHM10]. For Fano varieties, we instead know that $-K_X$ is ample, and do not have all of the nice results of the MMP at our disposal. How can we construct limits of families of Fano varieties in a functorial way?

Example 3.4. We also usually want our moduli spaces to be separated; i.e. "families have unique limits." Here, we encounter a problem: the moduli space of slc Fano varieties with fixed volume and dimension is not separated. (Compare to: moduli of varieties with ample canonical divisor, where it is separated.)

For example, let $X = \mathbb{P}^1$. Then, X is Fano and isotrivially degenerates to $X_0 = \mathbb{P}^1 \cup \mathbb{P}^1$ where the two curves are glued at one point; i.e. we can take a family of smooth conics (which are all isomorphic) degenerating to $xy = 0$. The normalization of X_0 is $(\mathbb{P}^1, \Delta) \cup (\mathbb{P}^1, \Delta)$ where Δ is the conductor; one point on each $\mathbb{P}^1$. Then, $\mathrm{vol}(-K_X) = 2$; $\mathrm{vol}(-K_{X_0}) = 2\mathrm{vol}(-K_{\mathbb{P}^1} - \Delta) = 2$.

Example 3.5. To actually construct moduli spaces of varieties, we usually: (1) bound our moduli problem in some way so that we can embed all of the varieties in question into a fixed projective space. Then, (2) use the Hilbert scheme from that projective space to construct the moduli space (because Hilbert schemes are 'nice'). Here, we encounter another problem: the set of log terminal Fano varieties with fixed volume and dimension is not necessarily bounded.

For example, we can construct an unbounded number of log terminal degenerations of $\mathbb{P}^2$, all of which have anticanonical volume 9. Let (a, b, c) be a solution to the Markov equation

$$a^2 + b^2 + c^2 = 3abc$$

where a, b, c are relatively co-prime, and consider the weighted projective space $\mathbb{P}(a^2, b^2, c^2)$. All solutions to the Markov equation are obtained by successively permuting or performing the *mutation* $(a, b, c) \mapsto (a, b, 3ab - c)$ starting from the minimal solution $(1, 1, 1)$. The first few triples in the Markov tree are

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & (1,13,34)\cdots \\
 & & & (1,5,13) \arrow[ur, no head] \arrow[dr, no head] & \\
 & & & & (5,13,194)\cdots \\
(1,1,1) \arrow[r, no head] & (1,1,2) \arrow[r, no head] & (1,2,5) \arrow[uur, no head] \arrow[ddr, no head] & & \\
 & & & & (5,29,433)\cdots \\
 & & & (2,5,29) \arrow[ur, no head] \arrow[dr, no head] & \\
 & & & & (2,5,29)\cdots
\end{tikzcd}$$

corresponding to the weighted projective spaces $\mathbb{P}^2$, $\mathbb{P}(1, 1, 4)$, $\mathbb{P}(1, 4, 25)$, ... Any well-formed weighted projective space with anticanonical volume 9 corresponds to one of these and furthermore, they all admit a smoothing to $\mathbb{P}^2$ (see exercises). There are infinitely many of these surfaces, they all have volume 9, and have log terminal singularities. (So, the moduli space here is unbounded and "infinitely" non-separated.)

Example 3.6. Finally, to get a well-behaved moduli space (a "good quotient" of the associated moduli stack), we also typically like to have that the automorphism groups of the elements parameterized by the moduli problem are at the very least reductive (even better: finite). In any case, we know this is not true for Fano varieties (reductivity or finiteness). Again, contrast with what happens for varieties with ample canonical divisor.

The moral of the previous set of examples is that it is probably hopeless to have a well-behaved moduli space of all Fano varieties analogous to that of the general type case. But, all of these problems are *solved* by restricting to only K-(semi/poly)stable Fano varieties.

Theorem 3.7 ([Jia20, Bir19])**.** *The set of K-semistable Fano varieties of dimension n and volume V form a bounded family.*

Theorem 3.8 ([BLX22, Xu20])**.** *K-semistability is an open condition in $\mathbb{Q}$-Gorenstein families.*

Theorem 3.9 ([LWX19, BX19, BHLLX21, LXZ22, XZ20])**.** *The moduli stack of K-semistable Fano varieties of dimension n and volume V is proper, and the moduli space of K-polystable Fano varieties of dimension n and volume V is projective.*

Theorem 3.10 ([ABHLX20])**.** *The automorphism group of a K-polystable Fano variety is reductive.*

This culminates in the K-moduli theorem:

Theorem 3.11. *There is an Artin stack of finite type $\mathcal{M}_{n,V}$ parameterizing families of K-semistable Fano varieties of dimension n and volume V and an associated projective good moduli space $M_{n,V}$ parameterizing K-polystable Fano varieties.*

In terms of the explicit issues raised above, we avoid the problem of degenerating to non-normal varieties (K-semistable implies log terminal, which implies normal) and the unbounded issue (other than $\mathbb{P}^2$ itself, all of the weighted projective spaces gives in the unbounded example are K-unstable), and have properness and reductive automorphism groups.

Remark 3.12. Everything we have said so far can be done for log Fano pairs (X, D)! Essentially, replace $-K_X$ with $-(K_X + D)$ in all of the definitions.

3.1. **Exercises.**

(1) Let $X = \mathbb{P}(p, q, r)$ be a weighted projective space (with p, q, r relatively co-prime) such that $(-K_X)^2 = 9$. Prove that $p = a^2$, $q = b^2$, and $r = c^2$ such that $a^2 + b^2 + c^2 = 3abc$.

(2) (a) Show that the singularities on $\mathbb{P}(a^2, b^2, c^2)$ where $a^2 + b^2 + c^2 = 3abc$ are $\mathbb{Q}$-Gorenstein smoothable (i.e. locally around each singularity, construct a smoothing).
(b) Show that there are no local-to-global obstructions to deforming $\mathbb{P}(a^2, b^2, c^2)$, so (a) together with the fact that $(-K_X)^2$ is constant in a $\mathbb{Q}$-Gorenstein family implies that $\mathbb{P}(a^2, b^2, c^2)$ is smoothable to $\mathbb{P}^2$.

(3) Prove that the general cubic surface is K-semistable using openness of K-semistability. (Hint: find one with many automorphisms, like the Fermat or $xyz = w^3$, and compute α_G.)

(4) Find a smooth Fano threefold that is K-semistable but not K-polystable. (Hint: find an isotrivial degeneration of a smooth Fano threefold to a K-polystable threefold.)

4. K-stability of singular Fano varieties and K-moduli of cubic surfaces

The goal of this section is to introduce more invariants related to the study of K-stability to completely determine several K-moduli spaces. We have already learned some explicit tools for "what K-stability is" and that a K-moduli space exists. But, how can we determine all of the objects in a particular K-moduli space? We will introduce one more powerful invariant that is particularly useful in this setting.

4.1. **Local-to-global principles and normalized volume.** We start with a motivational theorem:

Theorem 4.1 ([Fuj18, Liu18])**.** *Assume X is a K-semistable $\mathbb{Q}$ Fano variety of dimension n. Then,*

$$(-K_X)^n \leq (n+1)^n.$$

Furthermore, equality holds if and only if $X \cong \mathbb{P}^n$.

Proof. We prove only the first statement. Choose a smooth point $x \in X$ and let $Y = \mathrm{Bl}_x X$ be the blow up of the point x, with birational morphism $\mu : Y \to X$ and exceptional divisor $E \subset Y$.

By assumption and the valuative criteria (Theorem 1.31), we must have $\beta(E) \geq 0$, i.e.

$$A_X(E)(-K_X)^n \geq \int_0^\infty \mathrm{vol}(-K_X - tE)dt.$$

Furthermore, denote $\mu : Y \to X$ the blow up of a smooth subvariety $Z \subset X$ of codimension k contained in the smooth locus of X with exceptional divisor E. Then,

$$K_Y = \mu^*(K_X) + (k-1)E.$$

So, if we blow up a smooth point on a variety of dimension n,

$$K_Y = \mu^*(K_X) + (n-1)E$$

and therefore

$$A_X(E) = 1 + \mathrm{coeff}_E(K_Y - \mu^* K_X) = 1 + n - 1 = n.$$

To compute β, we can estimate $\mathrm{vol}(-K_X - tE) := \mathrm{vol}(\mu^*(-K_X) - tE)$. Assume for simplicity that $t \in \mathbb{Q}_{\geq 0}$. Take an integer $m \in \mathbb{Z}_{\geq 0}$ such that $mt \in \mathbb{Z}_{\geq 0}$. Then, by definition,

$$\mathrm{vol}(\mu^*(-K_X) - tE) = \lim_{m \to \infty} \frac{h^0(Y, \mathcal{O}_Y(m\mu^*(-K_X) - mtE))}{m^n/n!}.$$

We can estimate the number of global sections:

$$\mu_* \mathcal{O}_Y(m\mu^*(-K_X) - mtE) = \mathcal{O}_X(-mK_X) \cdot \mathfrak{a}_{mt}$$

where $\mathfrak{a}_{mt} := m_x^{mt} = \{f \in \mathcal{O}_{x,X} \mid \mathrm{ord}_E(f) \geq mt\}$ is the (power of the) maximal ideal of the point x we blew up. Therefore,

$$\begin{aligned} h^0(Y, \mathcal{O}_Y(m\mu^*(-K_X) - mtE)) &= h^0(X, \mu_* \mathcal{O}_Y(m\mu^*(-K_X) - mtE)) \\ &= h^0(X, \mathcal{O}_X(-mK_X) \cdot \mathfrak{a}_{mt}) \\ &\geq h^0(X, \mathcal{O}_X(-mK_X)) - \mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt}). \end{aligned}$$

The last inequality comes from the exact sequence

$$0 \to \mathfrak{a}_{mt} \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{a}_{mt} \to 0$$

twisted by $\mathcal{O}_X(-mK_X)$ (which is locally free in a neighborhood of x, so isomorphic to $\mathcal{O}_X$ in a neighborhood of x, so twisting the third term in the sequence does nothing):

$$0 \to \mathcal{O}_X(-mK_X) \cdot \mathfrak{a}_{mt} \to \mathcal{O}_X(-mK_X) \to \mathcal{O}_X/\mathfrak{a}_{mt} \to 0.$$

The dimension of the global sections of the first sheaf is therefore bounded by the difference of the next two.

This implies that

$$\begin{aligned} \mathrm{vol}(-K_X - tE) &\geq \mathrm{vol}(-K_X) - \lim_{m \to \infty} \frac{\mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt})}{m^n/n!} \\ &= (-K_X)^n - \lim_{m \to \infty} \frac{\mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt})}{m^n/n!} \\ &= (-K_X)^n - \lim_{mt \to \infty} \frac{\mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt})}{m^n t^n/n!} \cdot t^n \\ &= (-K_X)^n - \mathrm{vol}(\mathrm{ord}_E) \cdot t^n \\ &= (-K_X)^n - t^n. \end{aligned}$$

We will encounter volumes of valuations (the term $\mathrm{vol}(\mathrm{ord}_E)$ in the previous equation) momentarily, but you can also compute the last few lines as an exercise: let $k = mt$, and prove that $\lim_{mt\to\infty} \dfrac{\mathrm{length}(\mathcal{O}_{x,X}/\mathfrak{a}_{mt})}{m^n t^n/n!} = 1$.

Finally, plugging this into the inequality from the β-invariant (which we know holds if X is K-semistable):

$$A_X(E)(-K_X)^n \geq \int_0^\infty \mathrm{vol}(-K_X - tE)dt,$$

we find that

$$n(-K_X)^n \geq \int_0^\infty \max\{(-K_X)^n - t^n, 0\}dt$$

so

$$n(-K_X)^n \geq \frac{n}{n+1}(-K_X)^n \sqrt[n]{(-K_X)^n}$$

or

$$(-K_X)^n \leq (n+1)^n.$$

$\square$

We can strengthen this inequality with a tool called the *normalized volume*. This is defined in terms of valuations, but for a new learner of the subject, you can think about divisors whenever you see valuations: divisors correspond to so-called divisorial valuations by taking a divisor E to the valuation ord_E.

Definition 4.2 ([ELS03]). Let $x \in X = \mathrm{Spec}R$ be a klt singularity and $v \in \mathrm{Val}_{x,X}$ be a valuation centered at x. The **volume** of v is

$$\mathrm{vol}(v) = \lim_{k\to\infty} \frac{\mathrm{length}(R/\mathfrak{a}_k)}{k^n/n!}$$

where $\mathfrak{a}_k = \{f \mid v(f) \geq k\}$.

There is also a definition of log discrepancy $A_X(v)$ for general valuations due to Jonsson and Mustaţă [JM12] which we will not discuss in detail. With these ingredients, we can define Li's normalized volume.

Definition 4.3 ([Li18]). With the above set up, the **normalized volume** is

$$\widehat{\mathrm{vol}}(v) := A_X(v)^n \cdot \mathrm{vol}(v)$$

and the **local volume** at x is

$$\widehat{\mathrm{vol}}(x, X) := \inf_{v\in\mathrm{Val}_{x,X}} \widehat{\mathrm{vol}}(v).$$

Remark 4.4. If V is a $\mathbb{Q}$-Fano variety, let $X = C(V, -rK_V)$ be the cone over V and and $x \in X$ the vertex of the cone. Because V is Fano, $x \in X$ is klt, and X has a partial resolution $\mu : Y \to X$ by blowing up the vertex with exceptional divisor $V_0 \cong V \subset Y$. Another definition of K-semistability of V is that

$$V_0 \text{ is a minimizer of } \widehat{\mathrm{vol}}(x, X).$$

So, this notion of normalized volume somehow also captures the stability.

Back to the inequality. Using the normalized volume, we have a *Local to Global* Theorem on the volume of K-semistable varieties.

Theorem 4.5 ([LL19]). *Let X be a K-semistable $\mathbb{Q}$ Fano variety. Then, for any $x \in X$,*

$$(-K_X)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\mathrm{vol}}(x, X).$$

Proof. The proof is the same as the proof of Theorem 4.1, keeping $\mathrm{vol}(\mathrm{ord}_E)$ and $A_X(E)$ as in their definitions (without replacing them by 1 and n). $\square$

This is a very *powerful* result, called a local-to-global theorem, because it relates the local invariants of the singularities (the normalized volume) to a global invariant (the anticanonical volume). So, it allows you to constrain what singularities can appear relative to the anticanonical volume and vice versa. We will use this to our advantage later!

Here are some properties of the local volume function $\widehat{\mathrm{vol}}(x, X)$:

Property 4.6. (1) [dFEM04, Li18] If X has dimension n and $x \in X$ is smooth, then

$$\widehat{\mathrm{vol}}(x, X) = n^n$$

(2) [LX19] If X has dimension n, then for any $x \in X$,

$$\widehat{\mathrm{vol}}(x, X) \leq n^n$$

and equality holds if and only if x is smooth. Combining this with Theorem 4.5 gives Theorem 4.1.

(3) [Liu18] If $x \in X = (0 \in \mathbb{A}^n/G)$ is a quotient singularity where $G \subset \mathrm{GL}_n(\mathbb{C})$ acts freely in codimension 1, then

$$\widehat{\mathrm{vol}}(x, X) = \frac{n^n}{|G|}.$$

If $x \in X$ is a quotient of an arbitrary variety by $G \subset \mathrm{GL}_n(\mathbb{C})$, then

$$\widehat{\mathrm{vol}}(x, X) \leq \frac{n^n}{|G|}.$$

Combining this with Theorem 4.5 gives: if X is a K-semistable Fano variety, then for any quotient singularity $x \in X = (0 \in \mathbb{A}^n/G)$,

$$(-K_X)^n \leq \frac{(n+1)^n}{|G|}.$$

(4) [LX19] If X has dimension $n = 2$ or $n = 3$ and $x \in X$ is **not** a smooth point, then

$$\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n$$

and equality holds if and only if x is an ordinary double point. Conjecturally, the ordinary double point *always* gives the second largest volume.

Definition 4.7. The singularity $\frac{1}{n}(a, b)$ is the surface quotient singularity obtained by the action $\mathbb{A}^2/\mu_n$, where a primitive root of unity ζ_n acts by $\zeta_n \cdot (x, y) = (\zeta_n^a x, \zeta_n^b y)$.

Example 4.8. Previously, you proved in the exercises that $\mathbb{P}(1, 1, 2)$ is K-unstable using the β-invariant. Let's prove it again using the normalized volume.

By Property 4.6(3), we know that if X is a K-semistable Fano variety with a quotient singularity $\mathbb{A}^n/|G|$, then $(-K_X)^n \leq \frac{n+1)^n}{|G|}$. Let's plug in the associated values for $\mathbb{P}(1, 1, 2)$:

- $n = 2$ (the dimension of X)
- Because X is a (singular) quadric surface in $\mathbb{P}^3$, by adjunction, $(K_{\mathbb{P}^3} + X)|_X = K_X$, so $\mathcal{O}_X(K_X) = (\mathcal{O}_{\mathbb{P}^3}(-2))|_X$, and therefore

$$K_X^2 = (\mathcal{O}_{\mathbb{P}^3}(-2)|_X)^2 = \mathcal{O}_{\mathbb{P}^3}(-2) \cdot \mathcal{O}_{\mathbb{P}^3}(-2) \cdot \mathcal{O}_{\mathbb{P}^3}(2) = 8.$$

Alternatively, you could compute $(-K_X)^2$ using intersection theory on weighted projective space: $\mathcal{O}(K_X) = \mathcal{O}(-1-1-2) = \mathcal{O}(-4)$, and $(-K_X)^2 = \frac{(-4)^2}{2} = 8$.

- The singularity on $\mathbb{P}(1,1,2)$ can be described as the quotient singularity $\frac{1}{2}(1,1)$. So, $|G| = |\mu_2| = 2$.

Now we plug in! We see that

$$(-K_X)^2 = 8 > \frac{3^2}{2} = \frac{9}{2}$$

so X is K-unstable.

See the exercises for practice with quotient singularities.

4.2. **K-moduli of cubic surfaces.** Now, let's use the normalized volume to do some moduli! In a moduli problem, we wish to classify the objects that can appear. Suppose X is a K-semistable object in some moduli space. Theorem 4.5 can often be used to give a bound on the index of $-K_X$, which is related to the singularities that can appear on X, and the various reformulations can give more precise statements.

Consider a smooth cubic surface X in $\mathbb{P}^3$. By adjunction, $\mathcal{O}(K_X) = \mathcal{O}_{\mathbb{P}^3}(-1)|_X$, so X is Fano and $(-K_X)^2 = 3$. In other words, cubic surfaces are examples of degree three del Pezzo surfaces.

Question 4.9. What does the moduli space $\mathcal{M}^{\mathrm{sm}}_{2,3}$ of K-(semi/poly)stable degree three del Pezzo surfaces look like? (I've put the superscript sm to indicate that we are only looking at *smoothable* surfaces; i.e. only the 'main' component of the moduli space.)

In the exercises, you show every smooth del Pezzo surface of degree 3 is a cubic surface. What about the singular ones? Suppose X is a K-semistable singular del Pezzo surface, and let $x \in X$ be a singular point. We know X is normal and log terminal by the K-semistable assumption. As, log terminal surface singularities are all quotient singularities, we can use Property 4.6(3) to bound the normalized volume. Write $(x \in X) = (0 \in \mathbb{A}^2/G)$.

Theorem 4.5 together with Property 4.6(3) says

$$(-K_X)^2 \leq \frac{9}{|G|}.$$

We know $(-K_X)^2 = 3$, and we are assuming $x \in X$ is not smooth (so $|G| > 1$) so this implies that

$$2 \leq |G| \leq 3.$$

In other words, $|G| = 2$ or $|G| = 3$. There are only three choices for the resulting singularity $x \in X$:

(1) $G = \mu_2$ and $x \in X$ is an A_1 (or $\frac{1}{2}(1,1)$) singularity, which is the quotient $\mathbb{A}^2/\mu_2$ where μ_2 acts by $-1 \cdot (x,y) = (-x,-y)$
(2) $G = \mu_3$ and $x \in X$ is an A_2 (or $\frac{1}{3}(1,2)$) singularity, which is the quotient $\mathbb{A}^2/\mu_3$ where a cube root of unity $\zeta_3 \in \mu_3$ acts by $\zeta_3 \cdot (x,y) = (\zeta_3 x, \zeta_3^{-1} y)$
(3) $G = \mu_3$ and $x \in X$ is a $\frac{1}{3}(1,1)$ singularity, which is the quotient $\mathbb{A}^2/\mu_3$ where a cube root of unity $\zeta_3 \in \mu_3$ acts by $\zeta_3 \cdot (x,y) = (\zeta_3 x, \zeta_3 y)$

But, by the classification of smoothable log terminal surface singularities (e.g. [KSB88, §3] or [Kol, §6.6]), the third choice in the list is **not** smoothable! So, $x \in X$ must be an A_1 or A_2 singularity. By Exercise 8, we know that A_n singularities are Gorenstein, so any K-semistable del Pezzo surface of degree 3 X is Gorenstein, so $-K_X$ is Cartier. In fact, once we know it is Cartier, it is very ample by a result of Fujita (this is true for cubics in any dimension–see [Fuj90]) so $|-K_X| : X \hookrightarrow \mathbb{P}^3$ as a (singular) cubic surface.

So far, we have shown:

Theorem 4.10. *If $[X] \in \mathcal{M}^{\mathrm{sm}}_{2,3}$ is a K-semistable $\mathbb{Q}$ Fano surface of degree 3, then X is a cubic surface in $\mathbb{P}^3$ with at worst A_1 or A_2 singularities.*

Now, we know that any element parameterized by $\mathcal{M}^{\mathrm{sm}}_{2,3}$ is really just a surface in $\mathbb{P}^3$. To determine the K-stability of such a thing, does that mean we are allowed to restrict to test configurations where the central fiber is also in $\mathbb{P}^3$? In other words, can we consider only one-parameter subgroups of PGL_4 in the test configuration definition?

Depending on your background, this *might* be ringing some sort of bell. If we have objects in $\mathbb{P}^n$, and degenerate along one-parameter subgroups of PGL_{n+1}, and compute some sort of weight of this action.... This looks just like GIT! This is true in this case.

Theorem 4.11 ([OSS16]). *GIT = K stability for cubic surfaces.*

Proof. (Sketch.) **Step 0: K stability $\implies$ GIT stability.**

First, we show K $\implies$ GIT (this is a general idea due to Paul and Tian [PT06] for hypersurfaces). *Basic idea: one parameter subgroups are test configurations, so if all of the test configurations have positive weight, then so should all the one-parameter subgroups.*

By assumption, if X is K-(semi)stable, we have $\mathrm{Fut}(\mathcal{X}, \mathcal{L})(\geq) > 0$ for any test configuration. And, we proved it is a hypersurface $X \subset \mathbb{P}^3$, so given any one-parameter subgroup $\lambda \subset \mathrm{PGL}_4$, this induces a test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$.

Paul and Tian [PT06] show that that the Futaki invariant is proportional to the GIT weight, i.e.

$$\mathrm{Fut}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) = a\mu^{\mathcal{O}(1)}([X], \lambda)$$

where $a > 0$ is a positive constant and $\mu^{\mathcal{O}(1)}$ is the GIT weight. Therefore, K-semistability of X implies that the GIT weight is ≥ 0 for every one-parameter subgroup, hence the Hilbert-Mumford criterion implies that X is GIT-semistable.

Now, we want to show GIT stability $\implies$ K stability: Suppose $X \subset \mathbb{P}^3$ is GIT polystable (the other cases are similar). We want to show that X is K-polystable.

Step 1: Openness of K-moduli. There exists a K-stable cubic surface (see the exercises, or use §2) and the K-stable locus is Zariski open, so the general one is K-stable.

Step 2: Properness of K-moduli. Take a smoothing $\mathcal{X} \to C$ over a pointed curve $0 \in C$ such that $\mathcal{X}_0 \cong X$ is the cubic surface we know is GIT polystable. The general fiber $\mathcal{X}_t$ is a smooth cubic surface and, from the previous step, we can assume $\mathcal{X}_t$ is K-stable. By properness of K-moduli, up to base change, there exits a family $\mathcal{X}' \to C$ such that $\mathcal{X}' \setminus \mathcal{X}'_0 \cong \mathcal{X} \setminus \mathcal{X}_0$ and $X' := \mathcal{X}'_0$ is K-polystable. In simpler terms, if $\mathcal{X}_0 = X$ is not K-polystable, we know there is some K-moduli polystable limit $\mathcal{X}'_0$, so we put that in our family instead.

Step 3: Local to Global Volume Comparison. From our work already using Theorem 4.5, because $\mathcal{X}'_0$ is K-polystable, it is a cubic surface. By Step 0, because K-polystability implies GIT-polystability, $\mathcal{X}'_0$ is a GIT polystable surface. But now, $X = \mathcal{X}_0$ and $\mathcal{X}'_0$ are two polystable limits of the same family of surfaces, so by separatedness of the GIT moduli space, we must have $X \cong \mathcal{X}'_0$. Therefore, X is K-polystable. □

Corollary 4.12. Because all smooth cubic surfaces are GIT stable, this implies that all smooth cubics are K stable.

Using the index bound from the normalized volume, a similar result is true in higher dimensions:

Theorem 4.13 ([LX19, Liu22]). *GIT = K stability for cubic threefolds and cubic fourfolds.*

This is expected to hold in higher dimensions, and would follow from the conjectural Property 4.6(4) in higher dimensions.

Conjecture 4.14. *GIT = K stability for cubic hypersurfaces.*

4.3. **Exercises.**

(1) Prove that there are no nontrivial K-semistable degenerations of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (show any degeneration must be smooth using the local volume, and use rigidity of smooth Fanos).

(2) The singularities $\frac{1}{4}(1,1)$ and $\frac{1}{4}(1,3)$ are smoothable, so could appear on K-semistable degenerations of del Pezzo surfaces. What is the maximal degree of a del Pezzo surface for which they could appear? Bonus: exhibit a degree d del Pezzo surface with at least one of these singularities.

(3) Prove that any weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ not equal to $\mathbb{P}^n$ is K-unstable.

(4) In the proof of the index bound, we used the following:

If $x \in X$ is a smooth point (you may assume $x \in X = 0 \in \mathbb{A}^n$), show that

$$\lim_{k\to\infty} \frac{\operatorname{length}(\mathcal{O}_{x,X}/\mathfrak{a}_k)}{k^n/n!} = 1.$$

Prove this.

(5) If (X, cD) is a K-semistable log Fano pair, then the index bound inequality is:

$$(-K_X - cD)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\operatorname{vol}}(x, X, cD).$$

If $x \in X$ is a quotient singularity by a group G and $x \notin D$, then it is still true that

$$\widehat{\operatorname{vol}}(x, X, D) \leq \frac{n^n}{|G|}.$$

Suppose $X = \mathbb{P}(1,1,4)$ and $D \in \mathcal{O}_X(4d)$ for some integer d.

(a) If D passes through the singular point of X, show that it must have multiplicity at least 4 at the singular point, and that (X, cD) is K-unstable for any $c \in (0, \frac{3}{2d})$ by computing $\beta(E)$ where E is the exceptional divisor of the blow up of the singular point.

(b) If D does not pass through the singular point, use the index bound to prove that (X, cD) could only be K-semistable if $c \geq \frac{3}{4d}$.

(6) Let X be a degree d smooth del Pezzo surface. Prove that $-K_X$ is very ample and the linear system $|-K_X|$ embeds $X \hookrightarrow \mathbb{P}^d$ as a degree d surface.

(7) If X is a Gorenstein surface with ample $-K_X$ such that $(-K_X)^2 = 3$, prove that $|-K_X|$ is base point free and therefore very ample so X embeds in $\mathbb{P}^3$ as a cubic surface.

(8) Prove that an A_n singularity, the quotient $\mathbb{A}^2/\mu_n$ where μ_n acts by $\zeta_n \cdot (x,y) = (\zeta_n x, \zeta_n^{-1} y)$, is Gorenstein. (Hint/fact: any hypersurface singularity is Gorenstein, and the quotient of $\operatorname{Spec} k[x_1, \dots, x_n]$ by a finite group G is $\operatorname{Spec}(k[x_1, \dots, x_n]^G$, the ring of invariant poynomials under the group action G.)

(9) Prove that, in the minimal resolution of an A_n singularity, the exceptional divisor is a chain of smooth rational curves each with self intersection -2.

(10) Another quotient singularity is the quotient $\mathbb{A}^2/\mu_n$ where μ_n acts by $\zeta_n \cdot (x,y) = (\zeta_n x, \zeta_n y)$. These are often denoted by $\frac{1}{n}(1,1)$ singularities.

(a) The rational normal curve of degree n is defined as the image of the embedding of $\mathbb{P}^1 \to \mathbb{P}^n$ given by evaluation on the sections of $\mathcal{O}_{\mathbb{P}^1}(n)$ (i.e. $[x:y] \mapsto [x^n : x^{n-1}y : \cdots : y^n]$). Prove that the cone over this curve has a singularity of type $\frac{1}{n}(1,1)$.

(b) Prove that, for any n, the exceptional divisor of the minimal resolution of the cone over the rational normal curve of degree n is a single rational curve with self-intersection $-n$, and compute the discrepancy of the exceptional divisor.

(c) Prove that the cone over the rational normal curve of degree n is isomorphic to the weighted projective space $\mathbb{P}(1,1,n)$.

(d) Prove that $\mathbb{P}(1,1,n)$ is K-semistable if and only if $n = 1$.

5. Wall crossing for K-moduli spaces

In the last section, we will study wall-crossing phenomena for K-moduli spaces. Generally, it is interesting to study moduli as some coefficient is varying. With moduli of varieties, that often means we're looking at pairs (X, cD) and allowing the coefficient c to vary.

5.1. Wall crossings for moduli spaces of plane curves.

Example 5.1. Let's consider quartic curves in $\mathbb{P}^2$ as a motivating example. A quartic curve has genus 3 and any non-hyperelliptic genus 3 curve embeds as a quartic in $\mathbb{P}^2$.

Consider a compactification of moduli of pairs $(\mathbb{P}^2, cD)$, where D is a quartic plane curve. If $0 < c < \frac{3}{4}$, this is a log Fano pair, so we can construct a K-moduli space $\mathcal{M}_c$ of pairs for each $c \in \mathbb{Q}$.

Question 5.2. How do the K-moduli spaces change as c varies in the interval $(0, 3/4)$?

In fact, we can set this up much more generally for any log Fano pair (X, D) where $D \sim_{\mathbb{Q}} -rK_X$ for some $r \in \mathbb{Q}$, and ask the same question.

Theorem 5.3 ([ADL19])**.** *There are Artin stacks $\mathcal{M}_c$ (resp. good moduli spaces M_c) parameterizing K-semistable (resp. K-polystable) $\mathbb{Q}$-Gorenstein smoothable log Fano pairs (X, cD) with fixed Hilbert polynomial, $D \sim_{\mathbb{Q}} -rK_X$, X Fano, and $c \in (0, \min\{1, r^{-1}\})$.*

Furthermore, there are finitely many rational numbers

$$0 = c_0 < c_1 < c_2 < \cdots < c_k = \min\{1, r^{-1}\}$$

such that c-K-(poly/semi)stability conditions do not change for $c \in (c_i, c_{i+1})$. For each $1 \le i \le k-1$ and $0 < \epsilon \ll 1$, we have open immersions

$$\mathcal{M}_{c_i-\epsilon} \hookrightarrow \mathcal{M}_{c_i} \hookleftarrow \mathcal{M}_{c_i+\epsilon}$$

which induce projective morphisms

$$M_{c_i-\epsilon} \to M_{c_i} \leftarrow M_{c_i+\epsilon}.$$

Let's unpack this result in the case of degree d plane curves. We're considering moduli of pairs $(\mathbb{P}^2, cD)$ where $0 < c < 3/d$, D is a degree d curve, and trying to understand all K-semistable pairs of this form and their K-semistable degenerations.

We know that if (X, cD) is a K-semistable object in this moduli space, it is klt by Theorem 1.10, so X is log terminal. Additionally, Hacking and Prokhorov [HP10] prove that all possible X are those given in Example 3.5 and their partial smoothings. The possible singularities on X are all of the form $\frac{1}{n^2}(1, nl-1)$ where $\gcd(l, n) = 1$, and n is one of the elements of a Markov triple (a, b, c) satisfying $a^2 + b^2 + c^2 = 3abc$.

Definition 5.4. If $x \in X$ is a $\mathbb{Q}$-Gorenstein singularity, the **index** of x is the minimal positive integer m such that mK_X is Cartier near x.

The index of the singularity $\frac{1}{n^2}(1, na-1)$ is n.

Fact 2. Using the pairs version of Theorem 4.5, we get an index bound for the K-semistable pairs: if d is not divisible by 3,

$$\mathrm{ind}(x) \le \min\left\{\lfloor \frac{3}{3-dc} \rfloor, d\right\}.$$

Example 5.5. When $d = 4$ and $c < \frac{3}{4}$, this implies $\mathrm{ind}(x) \le 4$. By the description of the Markov triples in Example 3.5, only two triples have elements ≤ 4, so the only possible surfaces are $\mathbb{P}^2$ and $\mathbb{P}(1, 1, 4)$.

Also, from this index bound, we see that for $c \ll 1$, $\mathrm{ind}(x) = 1$, so X is Gorenstein. The only Gorenstein surface in Example 3.5 is $X = \mathbb{P}^2$. Therefore, for $c \ll 1$, all K-semistable pairs must be of the form $(\mathbb{P}^2, cD)$ for some degree d plane curve. Just as in the cubic surfaces example, now that we are working with a fixed projective space, we can try to relate the K-moduli spaces to GIT moduli spaces. Using similar ideas to those in Theorem 4.11, we can prove:

Theorem 5.6 ([ADL19])**.** *For $c \ll 1$, the K-moduli stack (space) parameterizing K-semi(poly)stable limits of pairs $(\mathbb{P}^n, cD)$, where D is a degree d hypersurface, is isomorphic to the GIT moduli stack (space).*

For low degree curves in $\mathbb{P}^2$, the GIT moduli space is well described. Therefore, to completely understand the K-moduli spaces, we can start with the GIT moduli space, increase the coefficient c until something "destabilizes" (which will give a wall crossing), find the K-semistable replacement, and continue.

Example 5.7. We will work this out completely for degree 4 curves. We know: for $c \ll 1$, the K-moduli space of pairs $(\mathbb{P}^2, cD)$ and their limits is isomorphic to the GIT moduli space. In other words, for $c \ll 1$ we start with GIT of quartic plane curves. We can 'guess' a value of c where the moduli space might change: from the index bound, when $c < \frac{3}{8}$, $\mathrm{ind}(x) < 2$, so we only have $\mathbb{P}^2$. But, when $c = \frac{3}{8}$, something else can happen!

In the GIT moduli space, there is a "special" point corresponding to the double conic. This point is special as it has the largest stabilizer group out of all GIT polystable points. And, when $c = \frac{3}{8}$, we can do the following. Consider a family D of smooth quartic curves degenerating to the double conic, inside $X = \mathbb{P}^2 \times \mathbb{A}^1$. In X, blow up the conic. This produces a threefold Y with exceptional divisor $E \cong \mathbb{F}_4$. Let D_Y be the strict transform of D in Y. Now, the surface that was the original central fiber of X is contractible, and we can contract it to produce a family Z of $\mathbb{P}^2$ degenerating to $\mathbb{P}(1,1,4)$. As we cross the wall at $c = \frac{3}{8}$, we can verify that the new central fiber $(\mathbb{P}(1,1,4), (c+\epsilon)D')$ is K-semistable. This is illustrated in Figure 1.

After we cross this wall, we have curves on both $\mathbb{P}^2$ or $\mathbb{P}(1,1,4)$ appearing. By the index bound, we know that these are the only surfaces that can appear in the K-moduli space for any c. In fact, for quartic curves, this is the only wall crossing in the K-moduli space! To see this, we use a result known as *interpolation.*

This version is as stated in [ADL19, Prop. 2.13] although these types of results were known before, see e.g. [Der16, Lemma 2.6] or [LS14].

Proposition 5.8. *Let X be a $\mathbb{Q}$-Fano variety. Let D_1 and D_2 be effective $\mathbb{Q}$-divisors on X satisfying the following properties:*

- *Both D_1 and D_2 are rational multiples of $-K_X$ under $\mathbb{Q}$-linear equivalence.*
- *$-K_X - D_1$ is ample, and $-K_X - D_2$ is nef.*
- *The log pairs (X, D_1) and (X, D_2) are K-(poly/semi)stable and K-semistable, respectively.*

Then we have

(1) If $D_1 \neq 0$, then $(X, tD_1 + (1-t)D_2)$ is K-(poly/semi)stable for any $t \in (0,1]$.
(2) If $D_1 = 0$, then $(X, (1-t)D_2)$ is K-semistable for any $t \in (0,1]$.

Furthermore, by [Oda13], a pair (X, D) with $-K_X - D \sim_{\mathbb{Q}} 0$ is K-semistable if and only if it is slc. Therefore, one corollary of such a result is:

Corollary 5.9. *Suppose (X, cD) is a log Fano pair such that $D \sim -rK_X$. If $(X, c_0 D)$ is K-(poly/semi)stable for some $c_0 \leq r^{-1}$ and the log canonical threshold $\mathrm{lct}(X, D) \geq r^{-1}$, then (X, cD) is K-(poly/semi)stable for any $c \in (c_0, r^{-1})$.*

Proof. Apply Proposition 5.8 for $D_1 = c_0 D$ and $D_2 = r^{-1} D$, using [Oda13] to say (X, D_2) is K-semistable. □

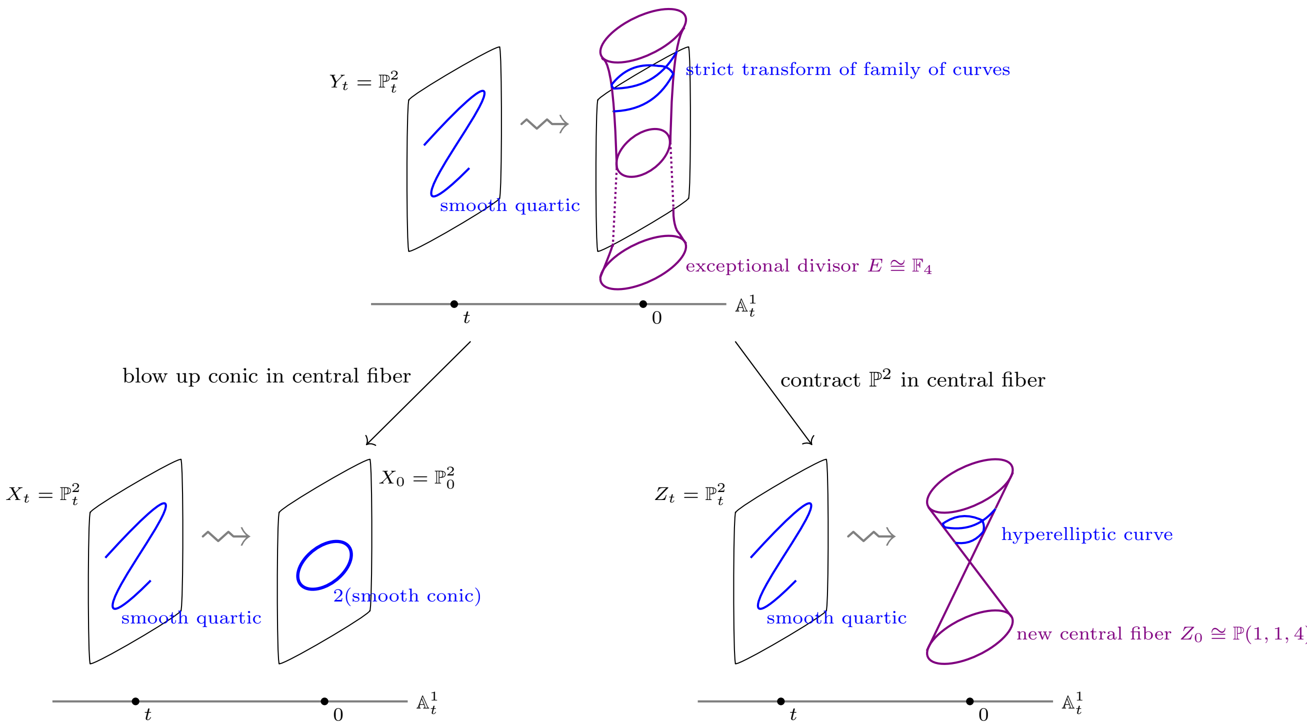

FIGURE 1. Replacement of the double conic.

For quartic curves, part (2) of Proposition 5.8 and the given corollary says that as long as the log canonical threshold of the pair $(\mathbb{P}^2, D)$ is at least $\frac{3}{4}$, then $(\mathbb{P}^2, cD)$ is K-polystable for all $c \in (0, \frac{3}{4})$. Every curve D in the GIT moduli space other than the double conic has this property, so for D other than the double conic, the pair $(\mathbb{P}^2, cD)$ is K-polystable for all c. After the first wall crossing, we include curves on $\mathbb{P}(1,1,4)$, but one can show they all also have log canonical threshold at least $\frac{3}{4}$, so they are K-polystable for all $c \in (\frac{3}{8}, \frac{3}{4})$. Therefore, we know there are no other wall crossings.

Let us revisit the wall-crossing at $c = \frac{3}{8}$. There are several important themes to recognize: first, even though we are working in the log Fano region, we can often use MMP-type operations to find K-semistable replacements, and we typically find walls by looking for special loci that have large stabilizers or small log canonical thresholds. This is similar to what happens for moduli of varieties of general type! Secondly, we achieve the wall crossing via *MMP-type operations on the moduli spaces themselves.* For instance, in this case, we took a specific family of quartic plane curves degenerating to the double conic, and computed a blow-up and blow-down in this family. But, this is also a morphism on the level of moduli spaces! We had the GIT moduli space $M_{\frac{3}{8}-\epsilon}$, and to cross the wall at $\frac{3}{8}$, we actually just blow-up (via a particular weighted blow-up) the point in the moduli space corresponding to the double conic. The resulting variety, which a priori just has a new exceptional divisor, also has a *modular* meaning. That new exceptional divisor actually parameterizes all of the curves we get on $\mathbb{P}(1,1,4)$ replacing that double conic. All wall crossings for quartics are summarized in Figure 2.

Note the picture *also* includes values of $c \geq \frac{3}{4}$ in the log Calabi Yau and general type region; see Chapter 2 for more on the general type side. The general type side of the picture was worked out by Hassett in [Has99]. Note for $c > \frac{5}{6}$, we write $\overline{M}_3$ instead of the moduli space of pairs $(\mathbb{P}^2, cD)$ and their degenerations; it is true in this case that for $c \in (\frac{5}{6}, 1]$, the moduli spaces are isomorphic,

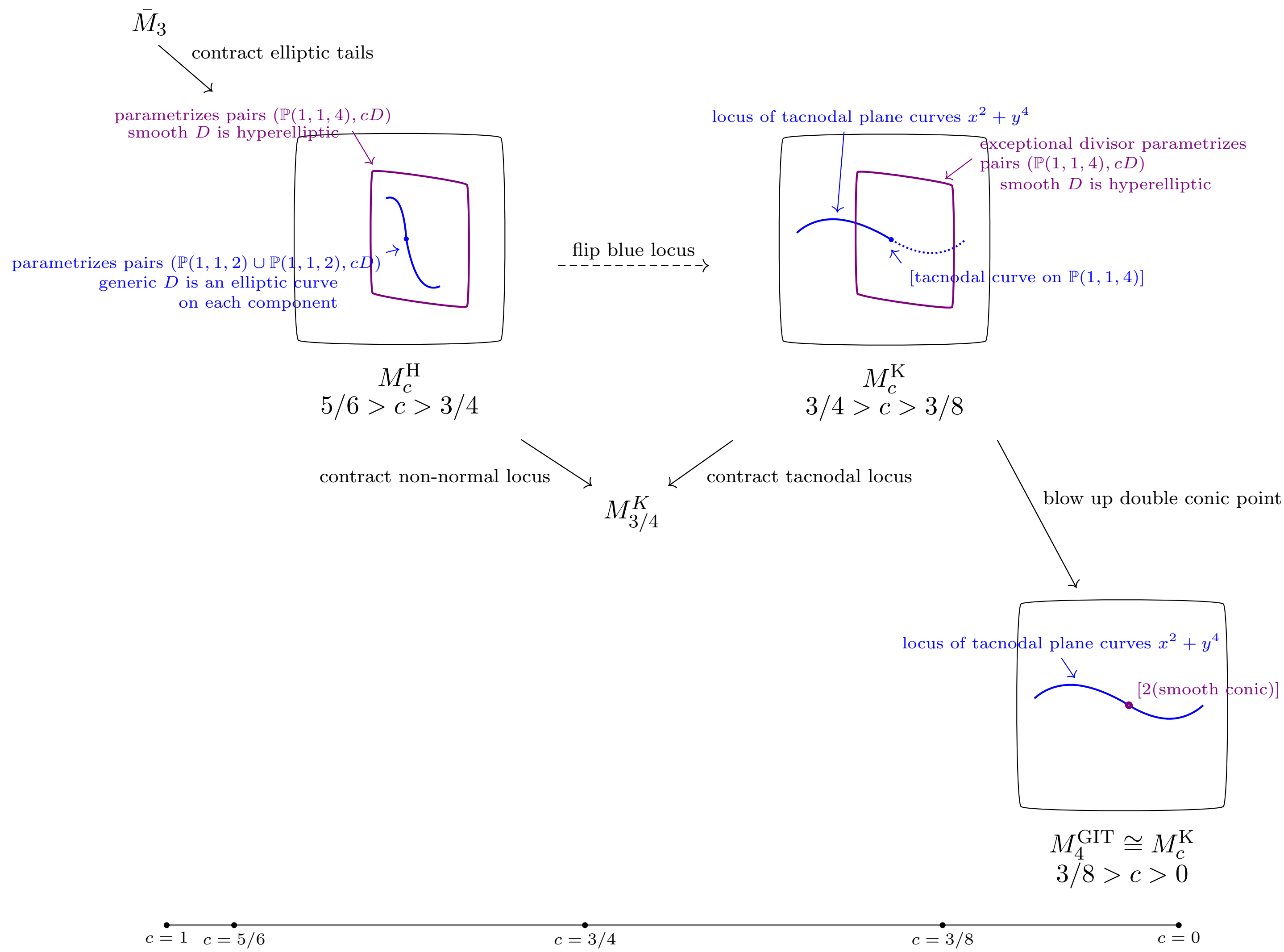

FIGURE 2. Wall crossings for moduli of quartic curves.

and at $c = 1$ the forgetful map $(X, D) \to D$ is also an isomorphism. In particular, the moduli space of pairs in this range is isomorphic to the moduli space of stable genus 3 curves [Has99]. Finally, at $c = \frac{3}{4}$, there does indeed exist a 'log Calabi Yau' moduli space but we will not go into detail on that here.

Let's do one more example studying the K-moduli spaces for quintic curves. Wee can also describe all of the K-moduli wall crossings; there are five of them (see [ADL19]). The first wall crossing is similar to the example above for quartics, so below we describe the second wall.

Example 5.10. There is a unique quintic curve C_0 with a singularity analytically of the form $x^2 = y^{13}$. This is (in a precise sense) the most singular reduced curve that can appear as a quintic, and it is GIT polystable. But, the pair $(\mathbb{P}^2, cC_0)$ must destabilize in K-moduli at some point before $c = \frac{3}{5}$: the log canonical threshold is $\frac{1}{2} + \frac{1}{13} < \frac{3}{5}$, and any K-semistable log Fano pair (X, cD) must be klt. So, at some point at or before the log canonical threshold of this curve, it must be replaced in the K-moduli space of pairs $(\mathbb{P}^2, cD)$ and their degenerations.

We will compute the birational transformations giving the wall crossing. Take a family of pairs $(\mathbb{P}^2, C_t)$, where C_t is a smooth plane quintic curve, degenerating to the singular curve $(\mathbb{P}^2, C_0)$ in a threefold $\mathbb{P}^2 \times \mathbb{A}^1_t$. The divisor C on this threefold has C_t smooth curves in the general fibers and C_0 in the central fiber. In the local coordinates (x, y, t) giving the parametrization of the quintic $x^2 = y^{13}$ in the central fiber $t = 0$, perform a $(13, 2, 1)$-weighted blow-up of the threefold. Let $\pi : X \to \mathbb{P}^2 \times \mathbb{A}^1$ be this blow up. This creates an exceptional divisor $\mathbb{P}(1, 2, 13)$ glued to $\tilde{\mathbb{P}}$, the strict transform of the central fiber $\mathbb{P} := \mathbb{P}^2 \times \{0\}$.

Then, the strict transform $\tilde{C_0}$ of the quintic curve in the central fiber is a -1 curve: we can show that $\pi|^*_{\tilde{\mathbb{P}}} C_0 = \tilde{C_0} + 26E$, where $E \cong \mathbb{P}(2, 13) \cong \mathbb{P}^1$, and $\tilde{C_0} \cdot E = 1$. So, intersecting with $\tilde{C_0}$, we get $25 = \tilde{C_0}^2 + 26$, so $\tilde{C_0}^2 = -1$.

With some thought to the cone of curves of the threefold X, one can show that the normal bundle $\mathcal{N}_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, so we can flop $\tilde{C_0}$ via the Atiyah flop. So, we perform a flop of the curve $X \dashrightarrow X^+$ (flopping it 'out' of $\tilde{\mathbb{P}}$ and 'in' to $\mathbb{P}(1, 2, 13)$). After this birational modification, the resulting image of $\tilde{\mathbb{P}}$ is contractible, and we contract that surface to a point to get a normal central fiber of this family. (Additionally, it is possible to show that the new curve in the central fiber is hyperelliptic, and that every hyperelliptic genus 6 curve can arise in this way!)

After these modifications, the resulting central fiber is a normal surface with a $\frac{1}{25}(1, 4)$ singularity with a smooth curve, and one can show it is K-semistable. This wall crossing is illustrated in Figure 3.

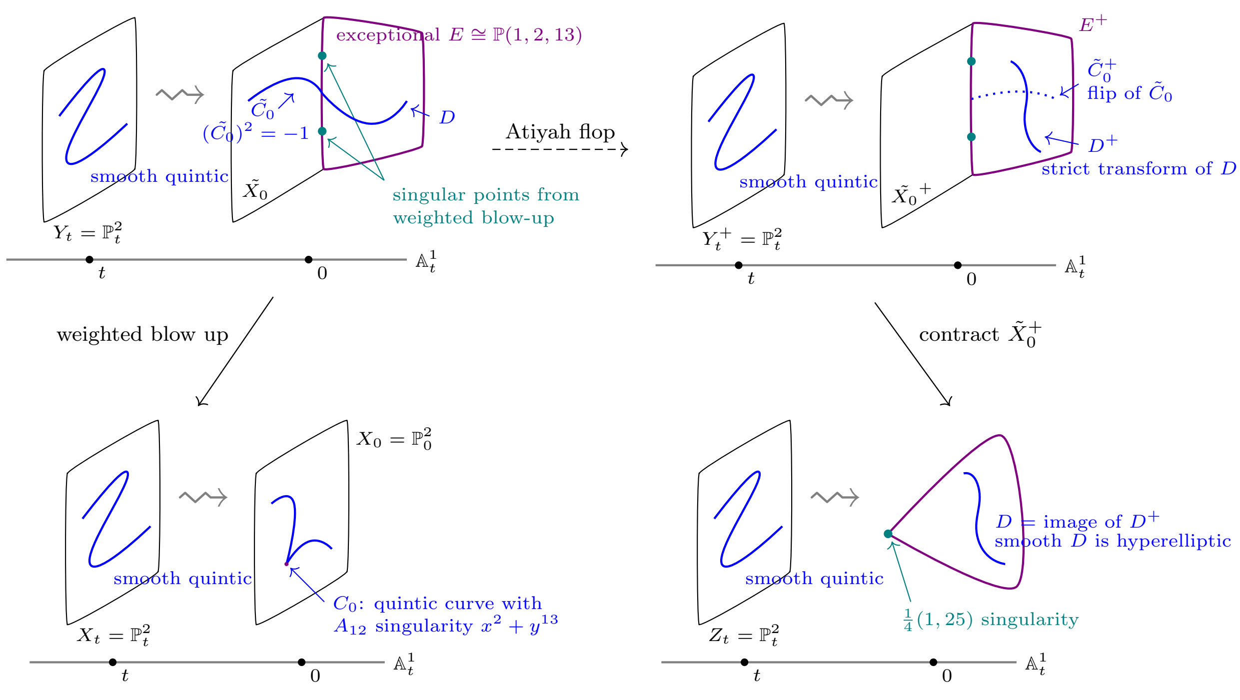

FIGURE 3. Replacement of the A_{12} quintic curve.

Details of the previous example are left to the exercises.

One final 'big-picture' reason to study these different moduli spaces is that using wall crossing for K-moduli spaces allows us to compare a priori very different moduli spaces. For example, a degree 2 K3 surface is naturally a double cover of $\mathbb{P}^2$ branched along a sextic curve. In this case, the

Baily-Borel moduli space $\mathcal{P}^*$ admits a natural $\mathbb{Q}$-factorialization $\widehat{\mathcal{P}} \to \mathcal{P}^*$, constructed by Shah and Looijenga. One could ask how these spaces compare to K-moduli spaces of sextic plane curves. In fact, we can interpolate between them using K-stability! Starting from $\mathcal{M}_c = \mathcal{M}^{GIT}$ the K-moduli space of sextic curves for $0 < c < 1/4$, we can identify $\widehat{\mathcal{P}} \cong \mathcal{M}_c$ with the K-moduli space for $1/4 < c < 1/2$ (after the first wall crossing, constructed similarly to that above). Then, the final space $\mathcal{P}^*$ is the ample model of the Hodge line bundle on $\mathcal{M}_{1/2-\epsilon}$. This is worked out in [ADL19].

Similarly, [ADL23a, ADL23b] show a similar phenomenon–interpolating between GIT and Baily-Borel using K-moduli–for degree 4 hyperelliptic K3s as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ by considering K-moduli wall crossings for pairs $(\mathbb{P}^1 \times \mathbb{P}^1, cD)$ where $D \in |\mathcal{O}(4,4)|$ and degree 4 K3s in general by considering K-moduli wall crossings for pairs $(\mathbb{P}^3, cS)$ where $S \in |\mathcal{O}(4)|$.

5.2. Exercises.

(1) Give an explicit description of a degeneration of $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$.

(2) Prove that the log canonical threshold of a singularity of the form $x^a = y^b$ ($a, b \in \mathbb{Z}^{>1}$) is $\frac{1}{a} + \frac{1}{b}$. (This could be a good time to investigate *weighted blow ups* instead of regular blow ups.)

(3) Show that the weighted projective curve $\mathbb{P}(a,b)$ is isomorphic to $\mathbb{P}^1$.

(4) Prove that $(\mathbb{P}^2, cL)$, where L is a line, is K-unstable for every $c \in (0,1)$. (This shows that the K-moduli space of lines in $\mathbb{P}^2$ is empty.)

(5) Prove that $(\mathbb{P}(1,1,4), cD)$ where D is the hyperplane section at infinity ($z = 0$ in the coordinates $[x : y : z]$ on $\mathbb{P}(1,1,4)$) could be K-semistable only if $c = \frac{3}{4}$. You may want 5.5.(b) above. Prove that it is in fact K-semistable. (This is how we get the first wall crossing in moduli of plane curves.)

(6) Prove that $(\mathbb{P}(1,1,4), cD)$ is K-unstable for every c if the multiplicity of D at the singular point of $\mathbb{P}(1,1,4)$ is at least four. Prove that it may be K-semistable for some c if the multiplicity is only two.

(7) This problem outlines some of the steps in finding the second wall for K-moduli of quintic curves.
   (a) Prove that there exists a quintic curve with an A_{12} singularity (locally, $x^2+y^{13}+h.o.t.$). Possible hint: how many terms do you need to vanish in the Taylor series expansion at the point $(0,0)$? How many parameters do quintics depend on?
   (b) Prove that the normalization of such a curve is rational.
   (c) Let D be a curve as in (a). Consider the pair $(\mathbb{P}^2, cD)$. Prove that if this pair is K-semistable, then $c \leq \frac{8}{15}$. Possible hint: do the $(13,2)$ weighted blow-up in the analytic coordinates (x,y) where the curve is $x^2 + y^{13}$ and let E be the exceptional divisor. Show that $\beta(E) \geq 0$ if and only if $c \leq \frac{8}{15}$.

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