Introduction to the Work of Caucher Birkar

Fields Medal Symposium Public Opening

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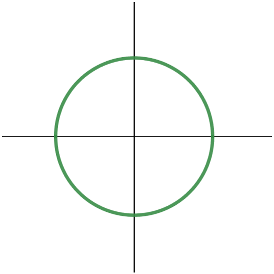
Introduction

We honor Caucher Birkar *"for his proof of the boundedness of Fano varieties and contributions to the minimal model program."*

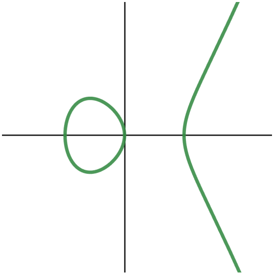
Algebraic Geometry

Algebraic geometry is the study of geometric objects defined by polynomials.

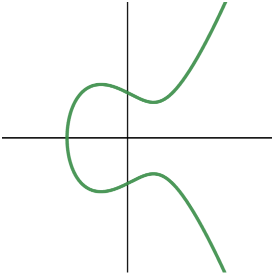
Examples



$x^2 + y^2 = 1$



$y^2 = x^3 - x$

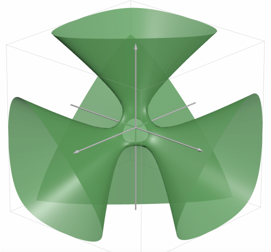


$y^2 = x^3 - x + 1$

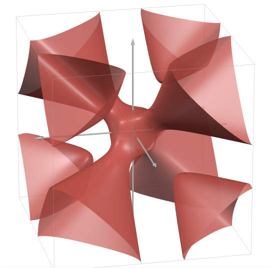
Algebraic Geometry

Algebraic geometry is the study of geometric objects defined by polynomials.

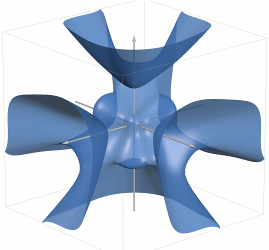
Examples



$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$



$$\begin{gathered} x^4 + y^4 + z^4 + \left(x^2 + y^2 + z^2\right) \\ -2\left(x^2y^2 + x^2z^2 + y^2z^2\right) + 2xyz = 1 \end{gathered}$$



$$\begin{gathered} x^5 + y^5 + z^5 - 8\left(x^3 + y^3 + z^3\right) - 2(x^3y^2 + x^3z^2 + y^3z^2 \\ +x^2y^3 + x^2z^3 + y^2z^3) - (x^2y^2 + y^2z^2 \\ +x^2z^2) - \left(x^2 + y^2 + z^2\right) + 4 = 0 \end{gathered}$$

Algebraic Geometry

What types of solutions do we allow?

Example

Quadratic equations usually have two solutions:

$$\begin{aligned} x^2 + 3x + 2 &= 0 \\ (x+1)(x+2) &= 0 \\ x &= -1, -2 \end{aligned}$$

but sometimes have none (if we only use real numbers):

$$x^2 + 1 = 0$$

Algebraic Geometry

What types of solutions do we allow?

Example

Quadratic equations usually have two solutions:

$$\begin{aligned} x^2 + 3x + 2 &= 0 \\ (x+1)(x+2) &= 0 \\ x &= -1, -2 \end{aligned}$$

~~but sometimes have none (if we only use real numbers):~~ and always do if we allow *complex numbers*:

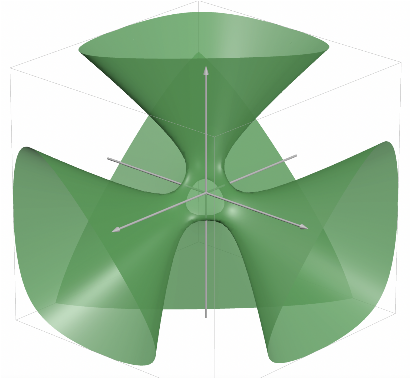
$$\begin{aligned} x^2 + 1 &= 0 \\ (x-i)(x+i) &= 0 \\ x &= i, -i \end{aligned}$$

We will always work with complex numbers so we have the 'expected' behavior of solutions to our polynomial equations.

Algebraic Geometry

Caution

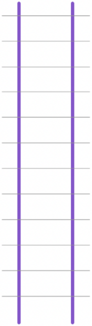
When we draw pictures, we can only draw the *real number* solutions, although we really work with the complex numbers.

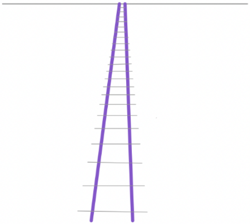


Algebraic Geometry

We will also add points "at ∞".

Like railroad tracks toward the horizon, two parallel lines in the xy-plane never intersect, but they do if they continue to infinity:





Classification of geometric objects

Fundamental Goal

Classify geometric objects defined by polynomial equations, where

- we allow the solutions to be complex numbers, and
- we add in extra points at ∞.

The Minimal Model Program

The minimal model program predicts that our geometric objects are made up of three building blocks:

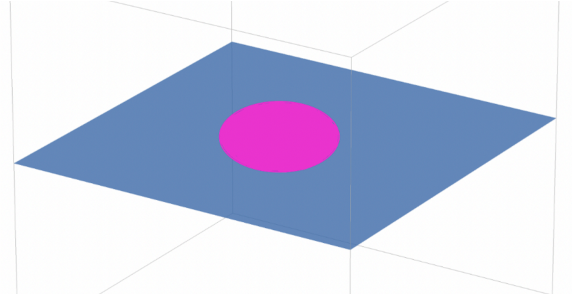
1. positively curved parts,
2. flat parts, and
3. negatively curved parts.

We expect that every geometric object can be modified to one that is built out of pieces with uniform curvature.

Curvature

Suppose you cut a small disk out of a flat piece of paper.

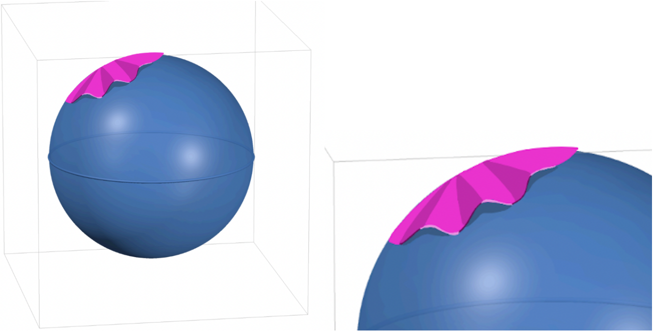
- On a flat surface, the disk fits on the surface without stretching or shrinking.



Curvature

Suppose you cut a small disk out of a flat piece of paper.

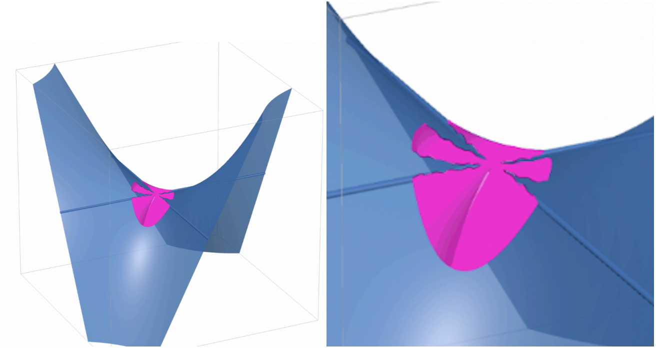
- On a sphere, the disk is too big to fit: you must shrink it to smoothly lay against the sphere.



Curvature

Suppose you cut a small disk out of a flat piece of paper.

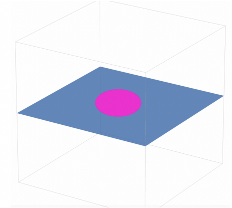
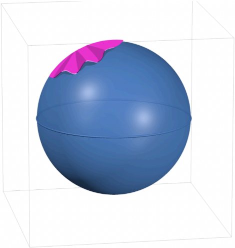
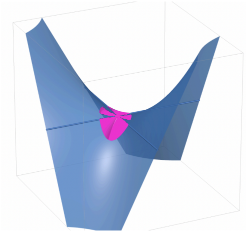
- On a saddle, the disk is too small to fit: you must stretch it or break it to smoothly lay against the surface.



Curvature

We can use this to define the *curvature* of a surface.

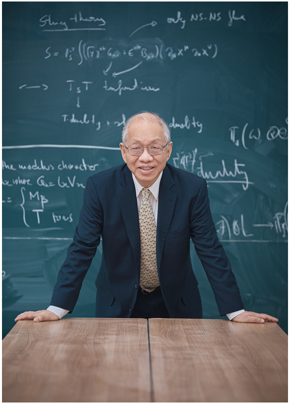
- If a small disk fits against the surface without changing the area, we say the surface is **flat**.
- If the area of the disk needs to be decreased to fit on the surface, we say it is **positively curved.**
- If the area of the disk needs to be increased to fit on the surface, we say it is **negatively curved.**

Curvature

In 1982, Yau received the Fields Medal for his proof of the *Calabi Conjecture*, among other significant contributions.

A consequence is that we have particular metrics (or measuring tapes) that we can use to define the volume of a ball and then the curvature of higher dimensional objects.



Curvature and the Minimal Model Program

We can define *curvature* in higher dimensions based on the volume of a small ball at each point and whether we need to increase or decrease the volume to fit on the shape.

Definition

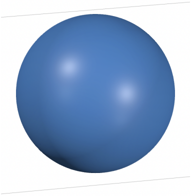
- If the volume needs to decrease, we say the geometric object is **Fano**.
- If the volume remains the same, we say the geometric object is **Calabi-Yau** or **K-trivial**.
- If the volume needs to increase, we say the geometric object is **canonically polarized** or of **general type**.

Caution

The way we measure volume in this context uses a different metric (measuring tape) than you may expect.

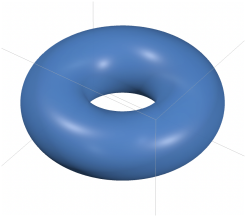
Curvature and the Minimal Model Program

Example



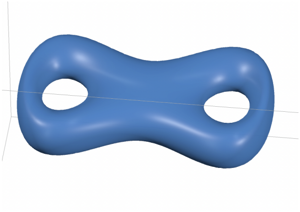
positively curved

Fano



flat

Calabi-Yau

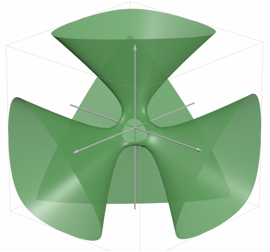


negatively curved

canonically polarized
or general type

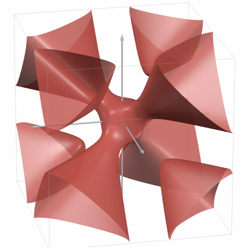
Curvature and the Minimal Model Program

Example



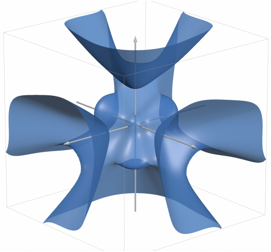
degree ≤ 3

Fano



degree $= 4$

Calabi-Yau



degree ≥ 5

canonically polarized
or general type

The Minimal Model Program

The minimal model program predicts that every geometric object defined by polynomial equations is made up of these three building blocks:

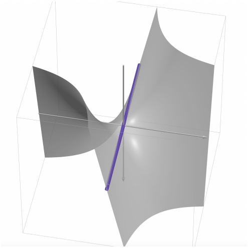
1. Fano objects,
2. Calabi-Yau objects, or
3. canonically polarized objects

The Minimal Model Program

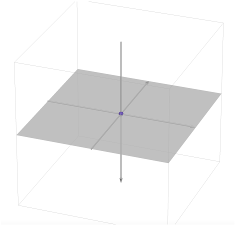
To run the MMP, we start with a geometric object, and try to sequentially modify it to be one of these three types.

For surfaces, we contract different curves until we reach something

1. canonically polarized (negatively curved everywhere), or
2. swept out by K-trivial (flat) curves, or
3. swept out by Fano (positively curved) curves.



before contraction



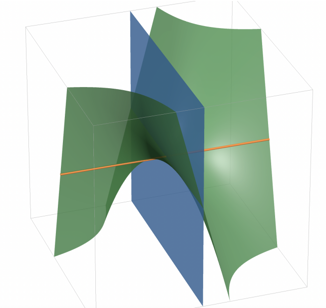
after contraction

The Minimal Model Program

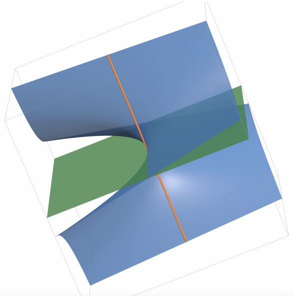
In three dimensions and above, we similarly want to contract different pieces until we reach something in the same list.

Problem

This is not always possible! We may need to perform surgery by cutting out a piece of the object and gluing it back in another way.



before surgery



after surgery

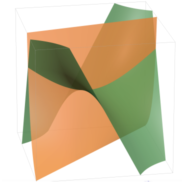
The Minimal Model Program

Main Conjecture

Starting with any geometric object defined by polynomial equations, we can sequentially modify it until we reach a new object that is:

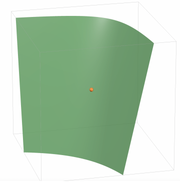
1. canonically polarized (negatively curved everywhere), or
2. swept out by K-trivial (flat) curves, or
3. swept out by Fano (positively curved) curves.

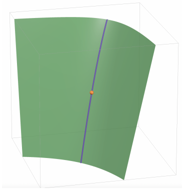
The Minimal Model Program



$\longrightarrow$

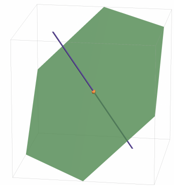
contract orange surface





$\longrightarrow$

surgery on purple line



The Minimal Model Program

In 1990, Mori received the Fields Medal for his work in the three-dimensional minimal model program.

Because of his work (also Kawamata, Kollár, Reid, Shokurov) it is known that the main conjecture holds in dimension 3.

The Minimal Model Program

Major Breakthrough: 2006

Theorem (Birkar, Cascini, Hacon, M^{c}Kernan)
The ‘surgeries’ (called flips) in the MMP algorithm are possible, and if we start with an object of *general type or Fano type*, then the main conjecture holds.

The purpose of this paper is to prove the following result in birational algebraic geometry:

Theorem 1.1. *Let (X, Δ) be a projective Kawamata log terminal pair.*
If Δ is big and $K_X + \Delta$ is pseudo-effective, then $K_X + \Delta$ has a log terminal model.

In particular, it follows that if $K_X + \Delta$ is big, then it has a log canonical model and the canonical ring is finitely generated. It also follows that if X is a smooth projective variety, then the ring

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is finitely generated.

Boundedness of Fano Varieties

- After BCHM, there still remain many open problems.
- For example, how many of each type of building block exist?

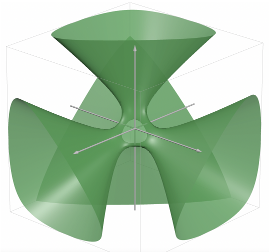
Borisov-Alexeev-Borisov (BAB) Conjecture

Fano shapes with mild singularities form a bounded family.

Boundedness of Fano Varieties

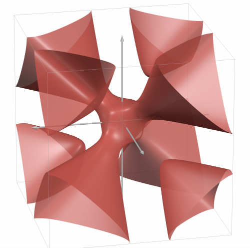
Boundedness says there are only ‘finitely many’ degrees of Fano objects, as is the case for surfaces in three-dimensional space:

Example



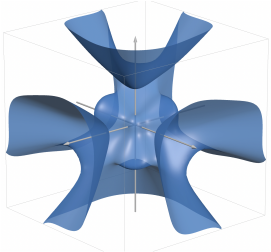
degree ≤ 3

Fano



degree $= 4$

Calabi-Yau

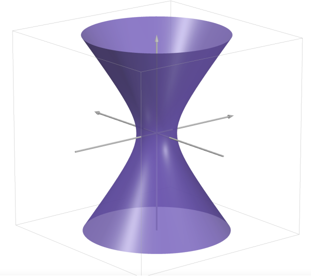


degree ≥ 5

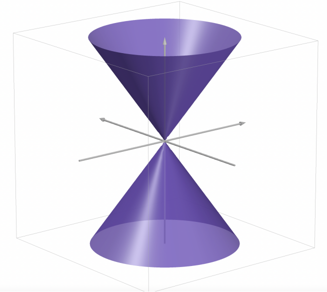
canonically polarized or general type

Boundedness of Fano Varieties

- In 1992, work of Kollár, Miyaoka, and Mori, showed that *smooth* Fano objects belong to a bounded family.



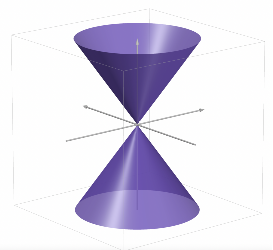
smooth (no corners or sharp points)

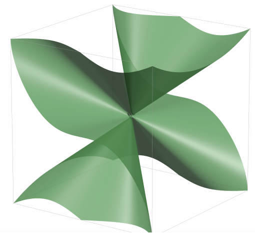


singular (not smooth)

Boundedness of Fano Varieties

- But, even in dimension 2, there exist unbounded families of singular Fano surfaces.





$$x^2 + y^2 = z^2$$

$$zy^2 = x^3 - xz^2 + z^3$$

Boundedness of Fano Varieties

If the singularities are controlled *just barely*, then:

BAB Conjecture

Fano geometric objects with mild singularities form a bounded family.

Major Breakthrough: 2016

Theorem (Birkar)
The BAB Conjecture is true.

The results mentioned above led Alexeev [1] and the Borisov brothers [9] to conjecture that, in any given dimension, Fano varieties with ϵ-lc singularities form a bounded family, for fixed $\epsilon > 0$. A generalised form of this statement, which is known in the literature as the Borisov-Alexeev-Borisov or the BAB conjecture, is our first result.

Theorem 1.1. *Let d be a natural number and ϵ be a positive real number. Then the projective varieties X such that*

- *(X, B) is ϵ-lc of dimension d for some boundary B, and*
- *$-(K_X + B)$ is nef and big,*

form a bounded family.

Conclusion

- Caucher Birkar has made tremendous strides in algebraic geometry and classification of algebraic varieties.
- Please join me in congratulating him on his incredible achievements!