COMPUTING SOME EXAMPLES OF BLOW-UPS

KRISTIN DEVLEMING

1. Introduction

This problem arose from studying families of pairs of *cameras*, projections $\phi_i : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from points $p_i \in \mathbb{P}^3$ for $i = 1, 2$, called the camera centers. These rational maps are resolved by blowing up the camera centers. We study a family of such maps over $\mathbb{A}^1$ where the camera centers are distinct for $t \neq 0$ but come together when $t = 0$ and the resolution of the family of maps.

2. Set-Up

Consider the family of rational maps $A_i : \mathbb{P}^3 \times \mathbb{A}^1 \dashrightarrow \mathbb{P}^2$ given by

$$A_1([x : y : z : w], t) = [x : y : z]$$

$$A_2([x : y : z : w], t) = [x : y : z + t]$$

The locus of indeterminacy of the map $(A_1, A_2) : \mathbb{P}^3 \times \mathbb{A}^1 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ is at least set-theoretically given by the union of the individual loci: $W = V(x, y, z) \cup V(x, y, z + t) = V(x, y, z(z + t))$.

To resolve the rational map, we let f be the composition of (A_1, A_2) with the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$, giving the rational map $f : \mathbb{P}^3 \times \mathbb{A}^1 \dashrightarrow \mathbb{P}^8$

$$f([x : y : z : w], t) = [x^2 : xy : x(z + t) : yx : y^2 : y(z + t) : zx : zy : z(z + t)]$$

From this, we see that, scheme-theorectically, the locus of indeterminacy is given by $Z = V(x^2, xy, x(z+t), y^2, y(z+t), xz, yz, z(z+t)) = V(x^2, y^2, xy, xz, yz, xt, yt, z(z+t))$. This is *not* the same as the previously mentioned locus $W = V(x, y, z(z + t))$. They coincide for $t \neq 0$, but for $t = 0$, $Z_0 = V(x^2, y^2, z^2, xy, xz, yz)$.

Ultimately, we'll blow up Z to resolve the map, noting that on each fiber of the map, this amounts to just blowing up the locus of indeterminacy of the fiber. This is clear for $t \neq 0$ and for $t = 0$, $Z_0 = V((x, y, z)^2)$, and the blow up of $\mathbb{P}^3$ at Z_0 isomorphic to the blow up of $\mathbb{P}^3$ at the point $(0, 0, 0)$.

For comparison, we'll blow up W and see that the rational map is not actually resolved.

3. Blowing Up W

On $\mathbb{P}^3$, $V(w)$ is completely contained in the domain of definition of f, so we only compute the blow up on $D(w)$. Abusing notation with the same coordinates, our map becomes $f : \mathbb{A}^3_{xyz} \times \mathbb{A}^1_t \dashrightarrow \mathbb{P}^8$

$$f((x, y, z), t) = [x^2 : xy : x(z + t) : yx : y^2 : y(z + t) : zx : zy : z(z + t)]$$

We then compute $X_W = Bl_W(\mathbb{A}^3 \times \mathbb{A}^1)$: because $W = V(x, y, z(z + t))$,

$$X_W = V(xT_2 - yT_1, xT_3 - z(z + t)T_1, yT_3 - z(z + 2)T_2) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{P}^2_{T_1T_2T_3}.$$

To resolve the map, we need to determine (locally) the defining equation of the exceptional divisor and divide the components of f by it. On $D(T_1)$, the blow-up has the equation

$$X_W = V(xT_2 - y, xT_3 - z(z + t)) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{A}^2.$$

Date: January 19, 2018.

The exceptional divisor E lives above W and has equation $V(x) \subset X_W$. Therefore, we take our rational map

$$f\left((x,y,z),t\right) = [x^2 : xy : x(z+t) : yx : y^2 : y(z+t) : zx : zy : z(z+t)]$$

and divide by x to get

$$\hat{f}\left(x,y,z,t,T_2,T_3\right) = [x : y : (z+t) : y : y^2/x : y(z+t)/x : z : zy/x : z(z+t)/x].$$

We can further simplify this using the equations of the blow up, since $y = xT_2$ and $z(z+t) = xT_3$:

$$\hat{f}\left(x,y,z,t,T_2,T_3\right) = [x : y : (z+t) : y : yT_2 : (z+t)T_2 : z : zT_2 : T_3].$$

One can check what happens on the other patches: on $D(T_2)$, E is given by $V(x)$, and on $D(T_3)$, E is given by $V(z(z+t))$. Doing the same computation (and projectivizing), we get an extension of f:

$$\hat{f}\left(x,y,z,t,[T_1 : T_2 : T_3]\right) = [xT_1 : yT_1 : (z+t)T_1 : yT_1 : yT_2 : (z+t)T_2 : z : zT_2 : T_3].$$

This does extend the rational map across any point of the original domain, but is still undefined along the curve $x = y = z = t = T_3 = 0$ (the curve in the exceptional divisor above the special fiber of the family $t = 0$ given by $T_3 = 0$).

4. Blowing Up Z

To completely resolve the map, we need to blow up the ideal of Z, which we do in the same way (working with the map $f : \mathbb{A}^3_{xyz} \times \mathbb{A}^1_t \dashrightarrow \mathbb{P}^8$). Because $Z = V(x^2, y^2, xy, xz, yz, xt, yt, z(z+t))$, we compute $X_Z = Bl_Z(\mathbb{A}^3 \times \mathbb{A}^1)$. The blow up is defined by equations that generate the kernel of the map

$$\begin{aligned} k[x,y,z,t][T_1,\ldots,T_8] &\to k[x,y,z,t] \\ T_1 &\mapsto x^2 \\ T_2 &\mapsto y^2 \\ T_3 &\mapsto xy \\ T_4 &\mapsto xz \\ T_5 &\mapsto yz \\ T_6 &\mapsto xt \\ T_7 &\mapsto yt \\ T_8 &\mapsto z(z+t) \end{aligned}$$

One then sees that

$$\begin{gathered} X_Z = V(y^2T_1 - x^2T_2, yT_1 - xT_3, zT_1 - xT_4, yzT_1 - x^2T_5, tT_1 - xT_6, ytT_1 - x^2T_7, z(z+t)T_1 - x^2T_8, \\ xT_2 - yT_3, xzT_2 - y^2T_4, zT_2 - yT_5, xtT_2 - y^2T_6, tT_2 - yT_7, z(z+t)T_2 - y^2T_8, \\ zT_3 - zT_4, zT_3 - xT_5, tT_3 - yT_6, tT_3 - xT_7, z(z+t)T_3 - xyT_8, \\ yT_4 - xT_5, tT_4 - zT_6, ytT_4 - xzT_7, (z+t)T_4 - xT_8, \\ xtT_5 - yzT_6, tT_5 - zT_7, (z+t)T_5 - yT_8, \\ yT_6 - xT_y, z(z+t)T_6 - xtT_8, z(z+t)T_7 - ytT_8, \\ T_1T_2 - T_3^2, T_4T_7 - T_5T_6, T_2T_4 - T_3T_5, T_3T_8 - T_4(T_5+T_7)) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{P}^7. \end{gathered}$$

Now, we can look locally to extend the map:

On $D(T_1)$, this simplifies to

$$X_Z = V(y^2 - x^2T_2, y - xT_3, z - xT_4, yz - x^2T_5, t - xT_6, yt - x^2T_7, z(z+t) - x^2T_8, T_2 - T_3^2, T_4T_7 - T_5T_6, T_2T_4 - T_3T_5, T_3T_8 - T_4(T_5 + T_7))$$

Then, our exceptional divisor has equation $E = V(x^2)$, so we can extend f to

$$\hat{f}\,(x, y, z, t, T_2, \ldots, T_8) = [1 : xy/x^2 : x(z+t)/x^2 : yx/x^2 : y^2/x^2 : y(z+t)/x^2 : zx/x^2 : zy/x^2 : z(z+t)/x^2]$$

and using the equations of the blow-up, we can write this as

$$\hat{f}\,(x, y, z, t, T_2, \ldots, T_8) = [1 : T_3 : T_4 + T_6 : T_3 : T_2 : T_5 + T_7 : T_4 : T_5 : T_8]$$

Similarly, we can check this on the other patches and projectivize to find one global extension

$$\hat{f}\,(x, y, z, t, [T_1 : \cdots : T_8]) = [T_1 : T_3 : T_4 + T_6 : T_3 : T_2 : T_5 + T_7 : T_4 : T_5 : T_8].$$

This is defined everywhere and one can check that it agrees with the given map f on its domain of definition.

Furthermore, for each fiber of the family, this map agrees with the fiber-wise resolutions of the map! One can see this by noting that the subvariety Z restricts on each fiber to the locus of indeterminacy for $t \neq 0$ and a power of it for $t = 0$.

5. One More Blow-up

For good measure, we also blow up the ideal (x, y, z, t) of the point $p = (0, 0, 0, 0)$ in the family and show that it does separate the lines $L_1 = V(x, y, z)$ and $L_2 = V(x, y, z + t)$:

$$X_p = Bl_p(\mathbb{A}^3 \times \mathbb{A}^1) = V(xT_2 - yT_1, xT_3 - zT_1, xT_4 - tT_1, yT^3 - zT_2, yT^4 - tT_2, zT_4 - tT_3).$$

One can check that the lines intersect the exceptional divisor on the patch $D(T_4)$:

$$X_p = Bl_p(\mathbb{A}^3 \times \mathbb{A}^1) = V(x - tT_1, y - tT_2, z - tT_3) \subset \mathbb{A}^3 \times \mathbb{A}^1 \times \mathbb{A}^3$$

The strict transform of the line L_1 is the closure of its preimage where $t \neq 0$, but for $t \neq 0$, the preimage of $V(x, y, z)$ is $V(T_1, T_2, T_3)$, so L_1 intersects the exceptional divisor E at $(T_1, T_2, T_3) = (0, 0, 0)$. Similarly for L_2, for $t \neq 0$, the preimage of $V(x, y, z + t)$ is $V(T_1, T_2, T_3 + 1)$, so the line L_2 intersects E at $(T_1, T_2, T_3) = (0, 0, -1)$. Therefore, in projective coordinates, L_1 intersects E at $[0 : 0 : 0 : 1]$ and L_2 intersects E at $[0 : 0 : -1 : 1]$.

Last but not least, in this blow up, we compute the extension of the map f:

$$\hat{f}\,((x, y, z), t, T_1, T_2, T_3) = [x^2/t : xy/t : x(z + t)/t : yx/t : y^2/t : y(z + t)/t : zx/t : zy/t : z(z + t)/t]$$

or

$$\hat{f}\,((x, y, z), t, T_1, T_2, T_3) = [xT_1 : yT_1 : (z + t)T_1 : yT_1 : yT_2 : (z + t)T_2 : zT_1 : zT_2 : (z + t)T_3].$$

This is still not defined along the exceptional divisor, but if we divide one more time by t we get something that is defined:

$$\tilde{f}\,((x, y, z), t, [T_1 : T_2 : T_3 : T_4]) = [T_1^2 : T_1T_2 : T_1(T_3+T_4) : T_1T_2 : T_2^2 : T_2(T_3+T_4) : T_1T_3 : T_2T_3 : T_3(T_3+T_4)].$$

This is undefined at the points $[0 : 0 : 0 : 1]$ and $[0 : 0 : -1 : 1]$ (which happen to be the intersection points of L_1 and L_2 with E).