Name: ______________________________

Stat 515
Midterm Examination II
April 6, 2010
(9:30 a.m. - 10:45 a.m.)

Instructions:

- The total score is 100 points.
- There are six questions. Each one is worth 20 points.
  TA will grade the best **five** questions that you solved.
- Show **ALL** your work!!
- Some questions have more than one parts. Check carefully to ensure that you don't miss any parts.
- Don't scratch on the line marked **Score** on the bottom of each page.
- You are allowed to use two 8.5x11(letter size) double-sided formula sheets and a calculator in this exam.

1. An oil prospector will drill a succession of holes in a given area to find a productive well. The probability that he is successful on a given trial is .2.

(10points) (a) Let Y is the number of trial on which the prospector will find the first productive well. What is the probability that the third hole drilled is the first to yield a productive well?

[Sol] Since Y is a geometric random variable with $p = 0.2$, $P(Y = 3) = (1 - 0.2)^2(0.2) = 0.128$.

(10points) (b) Suppose the prospector decide to drill exactly ten wells. Let X is the number of a productive well the prospector will find among ten wells. Then what is the probability that he will find three productive wells and seven unproductive wells?

[Sol] Since Y is a binomial random variable with $n = 10$ and $p = 0.2$, $P(Y = 3) = \binom{10}{3}0.2^3 0.8^7 = 0.2013$.

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2. Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. Let Y be the number of customers arriving per hour.

(5points) (a) What are the expected value of Y, $E(Y)$ and the variance of Y, $V(Y)$?

[Sol] Since Y has a Poisson distribution at an average of seven per hour, $E(Y) = \lambda = 7$. Then $Y \sim Poi(7)$.
$E(Y) = 7$ and $V(Y) = 7$.

(5points) (b) Find the probability that at least two customers arrive.

[Sol] $P(Y \geq 2) = 1 - P(Y < 2) = 1 - P(Y \leq 1) = 1 - e^{-7}7^0/0! - e^{-7}7^1/1!$

(10points) (c) Find the probability that exactly two customers arrive in the 2-hour period of time : between 4:30 p.m. and 6:30 p.m.

[Sol] Let X be the number of customers that arrive in a given two-hour period of time. Then X has a Poisson distribution with $14(= 2 * 7)$ and $P(X = 2) = \frac{14^2}{2!}e^{(-14)}$.

3. Let the cumulative distribution function(C.D.F.) of a continuous random variable Y be

$$F(y) = \begin{cases} 0 & y < 0, \\ \frac{y^2}{16} & 0 \le y < 4, \\ 1 & y \ge 4. \end{cases}$$

(7points) (a) Find the probability density function(p.d.f) of Y, $f(y)$.

[Sol]

$$f(y) \equiv \frac{\partial F(y)}{\partial y} = \begin{cases} \frac{y}{8} & 0 \le y < 4, \\ 0 & \text{elsewhere.} \end{cases}$$

(7points) (b) Find the expected value of Y, $E(Y)$.

[Sol] $E(Y) = \int_{-\infty}^{\infty} y f(y) dy = \int_0^4 y(y/8) dy = 8/3.$

(3points) (c) Find $P(1.5 \le Y \le 3)$.

[Sol] $P(1.5 \le Y \le 3) = F(3) - F(1.5) = \frac{3^2}{16} - \frac{1.5^2}{16} = 6.75/16.$

(3points) (d) Find $P(Y \ge 1.5 \mid Y \le 3)$.

[Sol] $P(Y \ge 1.5 \mid Y \le 3) = \frac{P(1.5 \le Y \le 3)}{P(Y \le 3)} = \frac{F(3) - F(1.5)}{F(3)} = \frac{6.75/16}{9/16} = 6.75/9.$

Score:____________

4. Answer the following questions.

(10points) (a) The operator of a pumping station has observed that demand for water during early afternoon hours has an exponential distribution with mean 100 cfs(cubic feet per second).
(i) Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.

[Sol] $P(Y > 200) = \int_{200}^{\infty} \frac{1}{100} e^{-y/100} dy = e^{-2} = 0.1353$

(ii) What water-pumping capacity should the station maintain during early afternoons so that the probability that demand will exceed capacity on a randomly selected day is 0.1?.

[Sol] $0.1 = P(Y > C) = \int_{C}^{\infty} \frac{1}{100} e^{-y/100} dy = e^{-C/100}$. Therefore, $C = 230.2585$.

(10points) (b) A candy maker produces mints that have a label weight of 20.4 grams. Assume that the weights of these mints have a normal distribution with mean 21.37 and variance 0.16 (i.e., the weights of these mints $\sim N(21.37, 0.16)$). Let Y denote the weight of a single mint selected at random from the production line.
(i) What is the probability that Y is larger or equal to 22.07?

[Sol] Since $Y \sim N(21.37, 0.16)$, $P(Y > 22.07) = P(Z > \frac{22.07-21.37}{0.4}) = P(Z > 1.75) = 0.0401$.

(ii) What is the appropriate value for C such that a randomly chosen single mint has a weight less than C with probability 0.8531?

[Sol] $P(Y < C) = P(Z < \frac{C-21.37}{0.4}) = 0.8531$. Then $P(Y < C) = 1 - P(Y > C) = 1 - P(Z > \frac{C-21.37}{0.4}) = 0.8531$ and $P(Z > \frac{C-21.37}{0.4}) = 0.1469$. So, $\frac{C-21.37}{0.4} = 1.05$ and $C = 21.79$.

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5. Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} ky_1y_2 & 0 \le y_1 \le 1, 0 \le y_2 \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(10points) (a) Find the value of k that makes $f(y_1, y_2)$ a joint probability density function.

[Sol] $1 = \int_0^1 \int_0^1 ky_1y_2dy_1dy_2 = k\int_0^1 \int_0^1 y_1y_2dy_1dy_2 = k/4$. Therefore, $k = 4$

(10points) (b) Find $F(0.2, 0.4)$.

[Sol] $F(0.2, 0.4) = P(Y_1 \le 0.2, Y_2 \le 0.4) = \int_0^{0.2} \int_0^{0.4} 4y_1y_2dy_1dy_2 = 0.0064$.

(10points) (c) Find $P(0.1 \le Y_1 \le 0.3, 0 \le Y_2 \le 0.2)$.
[Sol] $P(0.1 \le Y_1 \le 0.3, 0 \le Y_2 \le 0.2) = \int_{0.1}^{0.3} \int_0^{0.2} 4y_1y_2dy_1dy_2 = 0.0032$.

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6. Answer the following questions.

(6points) (a) Is the following statement is *true* or *false*? (No justification required)

Suppose Y is a continuous random variable. Then $P(a < Y \leq b) = P(a \leq Y < b)$ where a and b are real numbers with $a < b$.

[Sol] True

(8points) (b) Can the following serve as a cumulative distribution function (C.D.F.) of a random variable Y? Say why or why not(**Justification required**).

$$F(y) = \begin{cases} y & 0 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

[Sol] False. If $F(y)$ is a cdf, $F(-\infty) = 0$, $F(\infty) = 1$ and $F(y)$ should be nondecreasing.

(6points) (c) If the probability density function (p.d.f.) of a continuous random variable Y is

$$f(y) = \begin{cases} 1 & 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

then $P(Y = \frac{1}{2})$ is __0__(fill in the blank).(No justification required)

Score:__________