Homework 5/ Practice Midterm 1

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1. (a) Show that any integer can be written in the form $44X + 17Y$ where X and Y are integers.
   (b) Find integers x and y such that $44x + 17y = 100$.
2. Solve the system of congruences
$$x \equiv 1 \pmod 3,$$
$$x \equiv 2 \pmod 5,$$
$$x \equiv 3 \pmod 7.$$
(Hint: Use the first two congruences to find a single congruence $\pmod{15}$. Then combine this with the last congruence.)
3. (a) A common error in banking is to interchange two of the digits in an amount. Prove that the difference between the correct amount and the amount with the two digits interchanged is always divisible by 9.
   (b) A palindrome is a number that reads the same backward and forward, e.g. 1991, 23577532. Prove that a palindrome with an even number of digits is always divisble by 11.
4. Let a, b, c be positive integers.
   (a) Prove that $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$.
   (b) Prove that the Diophantine equation
$$aX + bY + cZ = 1$$
has a solution if and only if $\gcd(a, b, c) = 1$. (Hint: Use (a) to reduce the problem to two variables.)
5. (a) How many positive integers less than 101 have an *odd* number of positive divisors?
   (b) How many positive integers k have the property that
$$\operatorname{lcm}(6^6, 8^8, k) = 12^{12}?$$

Remark. For Problems 4 and 5 you may use (without proof) the following fact: if
$$m = p_1^{a_1} \dots p_r^{a_r}, n = p_1^{b_1} \dots p_r^{b_r}, \text{ and } k = p_1^{c_1} \dots p_r^{c_r},$$
then
$$\gcd(m, n, k) = p_1^{d_1} \dots p_r^{d_r} \text{ and } \operatorname{lcm}(m, n, k) = p_1^{e_1} \dots p_r^{e_r},$$
where $d_i = \min(a_i, b_i, c_i)$ and $e_i = \max(a_i, b_i, c_i)$.