MATH 704 – Spring 2010
Homework Set # 7

*Problem 54. Let $M \subset \mathbb{R}^3$ be a compact surface. Prove that there exists a point $p \in M$ where the Gaussian curvature $K > 0$.

**Hint:** Consider the map $F: M \rightarrow \mathbb{R}$, defined by

$$F(x, y, z) = x^2 + y^2 + z^2$$

and let $p$ be a point where $F$ attains a global maximum. Let $T_p(M) = p^\perp$ and prove that for every curve $\alpha(s) \subset M$, parametrized by arc length and such that $\alpha(0) = p$, $|\alpha''(0)| > 1/R$.

**Problem 55.** Let $M$ be a surface and $\Phi: U \subset \mathbb{R}^2 \rightarrow M$ a local parametrization. Let $X_u, X_v$ be the coordinate vector fields defined by $\Phi$. We say that $\Phi$ is an orthogonal parametrization if and only if $F = \langle X_u, X_v \rangle = 0$. As always denote by $E = \langle X_u, X_u \rangle$ and $G = \langle X_v, X_v \rangle$.

Prove that the Christoffel symbols are given by:

$$\Gamma^1_{11} = \frac{1}{2}(E_u/E) ; \quad \Gamma^2_{11} = -\frac{1}{2}(E_v/G) ; \quad \Gamma^1_{12} = \frac{1}{2}(G_v/G)$$

$$\Gamma^2_{12} = \frac{1}{2}(G_u/G) ; \quad \Gamma^1_{22} = -\frac{1}{2}(G_u/E) ; \quad \Gamma^2_{22} = \frac{1}{2}(E_v/E)$$

*Problem 56.** Let $M$ be a surface and $\Phi: U \subset \mathbb{R}^2 \rightarrow M$ an orthogonal parametrization. Prove that:

$$K = -\frac{1}{2\sqrt{EG}} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right).$$

*Problem 57.** Let $M$ be a surface and $\Phi: U \subset \mathbb{R}^2 \rightarrow M$ a local parametrization. Let $X_u, X_v$ be the coordinate vector fields defined by $\Phi$. We say that $\Phi$ is a conformal or isothermal parametrization if and only if $F = \langle X_u, X_v \rangle = 0$ and $E = G = \lambda(u, v)$.

a) Prove that the Gaussian curvature is given by:

$$K = -\frac{1}{2\lambda} \Delta(\log \lambda),$$

where $\Delta(f) = f_{uu} + f_{vv}$.

b) Compute the Gaussian curvature of the upper half plane endowed with the Poincaré metric:

$$\langle \partial_x, \partial_x \rangle = \langle \partial_y, \partial_y \rangle = 1/y^2 ; \quad \langle \partial_x, \partial_y \rangle = 0$$

**Problem 58.** Let $M$ be an open set in $\mathbb{R}^2$ with the Riemannian metric:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2(x, y) \end{pmatrix}$$

where $\lambda(x, y)$ is never zero on $M$. Compute the Christoffel symbols and the Gaussian curvature.

*Problem 59.** Compute the Christoffel symbols and the Gaussian curvature of the sphere $S^2$ in terms of spherical coordinates.
Problem 60. Compute the Christoffel symbols, and the Gaussian curvature of the sphere \( S^2 \) in terms of stereographic coordinates.

Problem 61. Let \( y = \varphi(x) \), \( x \in (a,b) \) be a \( C^\infty \) function such that \( \varphi(x) > 0 \) for all \( x \in (a,b) \). Let \( S \subset \mathbb{R}^3 \) be the surface of revolution obtained rotating the graph of \( \varphi \) around the \( x \)-axis. Let \( \Phi : (a,b) \times (0,2\pi) \to \mathbb{R}^3 \) be the local parametrization of \( S \) given by:

\[
\Phi(u,v) = (u, \varphi(u) \cos(v), \varphi(u) \sin(v))
\]

a) Compute \( E, F, G, e, f \) and \( g \) for the parametrization \( \Phi \).

b) Compute the Gaussian curvature of \( S \).

c) Compute the Christoffel symbols \( \Gamma^k_{ij} \) relative to the local coordinates defined by \( \Phi \).

d) Characterize the points of \( S \) at which the Gaussian curvature is positive, negative or zero.

e) Find the volume element of \( S \) (in terms of the local coordinates \( (u,v) \)).

*Problem 62. Repeat the previous problem for the surface of revolution obtained rotating the curve:

\[
C = \{(\varphi(u), \psi(u)) : \psi(u) > 0 \quad ; \quad \varphi'(u)^2 + \psi'(u)^2 = 1\}
\]

around the \( x \)-axis.

Problem 63. Find the surfaces of revolution (in the sense of the previous two problems) with constant Gaussian curvature.

*Problem 64. Compute the Gaussian curvature of the torus \( T \) obtained by rotating the circle:

\[
x^2 + (y - b)^2 = r^2 \quad ; \quad b > r
\]

around the \( x \)-axis. It is understood that the metric on \( T \) is induced from the Euclidean metric in \( \mathbb{R}^3 \).

*Problem 65. Let \((M,g)\) be a Riemannian manifold. Prove that given vector fields \( X,Y \in \mathcal{X}(M) \) the unique vector field \( \nabla_XY \) defined by the expression:

\[
2g(\nabla_XY, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y)
\]

is the Riemannian connection.

*Problem 66. Let \((M_1, g_1)\) and \((M_2, g_2)\) be Riemannian manifolds and let \( \nabla^{(1)} \) and \( \nabla^{(2)} \) denote the respective Riemannian connections. Let \( F : M_1 \to M_2 \) be an isometry.

a) Show that \( F_* \left( \nabla^{(1)}_X(Y) \right) = \nabla^{(2)}_{F_*X}(F_*Y) \quad ; \quad X,Y \in \mathcal{X}(M_1) \).

b) Let \( R_1 \) and \( R_2 \) denote the respective curvature tensors. Show that:

\[
R_1(X,Y,Z,W)(p) = R_2(F_*(X),F_*(Y),F_*(Z),F_*(W))(F(p))
\]
c) Let $\alpha: \mathbb{R} \to M_1$ be a $C^\infty$ curve and $Y$ a $C^\infty$ vector field along $\alpha$. Let $F_\ast(Y)$ denote the $C^\infty$ vector field along $F \circ \alpha$ defined by $F_\ast(Y)(F(\alpha(t))) = F_{\ast, \alpha(t)}Y(\alpha(t))$. Show that:

$$\frac{DF_\ast(Y)}{dt} = F_\ast\left(\frac{DY}{dt}\right)$$

d) Let $\alpha: \mathbb{R} \to M_1$ be a geodesic in $M_1$. Show that $F \circ \alpha$ is a geodesic in $M_2$.

*Problem 67.* Let $(M, g_M)$ be a Riemannian manifold and $\pi : M \to N$ a $C^\infty$ covering map.

a) Show that $N$ has a unique Riemannian metric $g_N$ for which $\pi$ is a local isometry.

b) Compute the Gaussian curvature of the torus $T$ with the Riemannian metric defined by the covering $\pi: \mathbb{R}^2 \to T$.

*Problem 68.* Let $(M, g)$ be a Riemannian manifold and let $\nabla$ denote the Riemannian connection. Given $\eta \in C^\infty(M)$, define a new Riemannian structure $g_\eta$ on $M$ by:

$$g_\eta(X, Y) = e^{2\eta}g(X, Y)$$

Let $\nabla^\eta$ denote the Riemannian connection associated to the metric $g_\eta$. Show that:

$$\nabla^\eta X Y = \nabla X Y + X(\eta Y) + Y(\eta X) - g(X, Y)\text{grad}(\eta)$$

where $\text{grad}(\eta)$ is the gradient of $\eta$ relative to the metric $g$.

*Problem 69.* Let $\nabla$ be a connection on a manifold $M$ and $\alpha \in \Lambda^r(M)$. Define:

$$(\nabla_X(\alpha))(Y_1, \ldots, Y_r) = X(\alpha(Y_1, \ldots, Y_r)) - \sum_{i=1}^r \alpha(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_r),$$

where $X, Y_1, \ldots, Y_r \in \mathcal{X}(M)$. Prove that:

a) $\nabla_X(\alpha) \in \Lambda^r(M)$.

b) $\nabla_{fX}(\alpha) = f \nabla_X(\alpha)$ for $f \in C^\infty(M)$.

c) Prove that $\nabla_X(f\alpha) = (\nabla_X f)\alpha + f\nabla_X(\alpha)$.

Generalize these statements to the case of a connection on a vector bundle.

*Problem 70.* Let $M$ be a manifold and let $\nabla$ denote a connection on $M$. Given $f \in C^\infty(M)$, we define a bilinear map

$$H_f: \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M) ; \quad H_f(X, Y) := (\nabla_X df)(Y)$$

a) Show that $H_f$ is $C^\infty(M)$-bilinear; i.e. a tensor on $M$.

b) Let $p \in M$ be a critical point of $f$, $u, v \in T_p(M)$ Prove that $H_f(u, v)$ agrees with the value $\text{Hess}_p(u, v)$ defined in Problem 45 of M703.

*Problem 71.* Let $(M, g)$ be a Riemannian manifold and let $\nabla$ denote the Riemannian connection. Let $\alpha \in \Lambda^1(M)$ and suppose $\nabla_X(\alpha) = 0$ for all $X \in \mathcal{X}(M)$. Show that $\alpha$ is closed.