MATH 704 – Spring 2010
Homework Set # 4

Only hand-in problems preceded by a *.

*Problem 25. Let $M$ be a manifold, $U$ an open subset, and $\alpha \in \Lambda^1(U)$. Define $\theta_\alpha: \mathcal{X}(U) \times \mathcal{X}(U) \to C^\infty(U)$ by

$$\theta_\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) ; \quad X, Y \in \mathcal{X}(U).$$

a) Prove that $\theta_\alpha$ is $C^\infty(U)$-bilinear and therefore it defines a section over $U$ of the bundle $\text{Bil}(TM)$ defined in the notes.

b) Show that $\theta_\alpha(X,Y) = -\theta_\alpha(Y,X)$.

c) Let $f \in C^\infty(U)$. Prove that $\theta_{f\alpha}(X,Y) = df(X)\alpha(Y) - df(Y)\alpha(X) + f\theta_\alpha(X,Y)$.

d) Suppose that in local coordinates

$$\alpha = \sum_{i=1}^n a_i dx_i ; \quad X = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} ; \quad Y = \sum_{j=1}^n c_j \frac{\partial}{\partial x_j}.$$  

Compute, explicitly, $\theta_\alpha(X,Y)$.

*Problem 26. Let $(M_1, g_1)$ and $(M_2, g_2)$ be Riemannian manifolds and let $F: M_1 \to M_2$ be a smooth map. We say that $F$ is an isometry if and only if

$$g_2(F(p))(F_* p(u), F_* p(v)) = g_1(p)(u,v)$$

for all $p \in M$ and all $u, v \in T_p(M)$.

a) Prove that if $F$ is an isometry then $F$ is a local diffeomorphism; i.e. for every $p \in M$ there exist open sets $U$ around $p$ and $V$ around $F(p)$ such that $F: U \to V$ is a diffeomorphism.

b) Let $M_1$ and $M_2$ be $n$-dimensional manifolds and $F: M_1 \to M_2$ a local diffeomorphism. Prove that given a Riemannian metric $g_2$ on $M_2$ there exists a unique Riemannian metric $g_1$ on $M_1$ making $F$ into an isometry.

c) Let $\pi: S^n \to \mathbb{P}^n$ be the natural projection and let $g_1$ be a Riemannian metric on $S^n$. Give a necessary and sufficient condition for the existence of a Riemannian metric $g_2$ on $\mathbb{P}^n$ making $\pi$ into an isometry. If $g_2$ exists, is it unique?

*Problem 27. Let $G$ be a Lie group. A Riemannian metric on $G$ is said to be left-invariant if the diffeomorphisms $L_g$ are isometries, for all $g \in G$.

a) Prove that every Lie group has a left-invariant metric.

b) Prove that a left-invariant metric is completely determined by the inner product it defines on the Lie algebra $T_e(G) \cong \mathfrak{g}$.
c) Find a left-invariant metric on $\mathbb{R}^3$ with the product defined by the Heisenberg product (See Problem 9)

d) Find a left-invariant metric on $\mathbb{R}^2$ with the product defined in Problem 13.

**Problem 28.** Let $M = U \subset \mathbb{R}^2$ be an open set and $\lambda \in C^\infty(M)$. Consider the Riemannian metric:

$$\langle \partial_x, \partial_x \rangle = \langle \partial_y, \partial_y \rangle = e^{\lambda(x,y)} ; \quad \langle \partial_x, \partial_y \rangle = 0.$$

Prove that it is possible to find local coordinates $(u_1, u_2)$ in $M$ such that

$$\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle = \delta_{ij}$$

if and only if $\lambda(x,y)$ is a harmonic function.

**Problem 29.** Let $\mathcal{H}$ be the upper-half plane, i.e.

$$\mathcal{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

We know from complex analysis that the partial linear transformations

$$(1) \quad F(z) = \frac{az + b}{cz + d} ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

are diffeomorphisms (in fact analytic automorphisms) of $\mathcal{H}$.

Consider the Poincaré metric on $\mathcal{H}$, that is the metric such that:

$$\langle \partial_x, \partial_x \rangle = \langle \partial_y, \partial_y \rangle = 1/y^2 ; \quad \langle \partial_x, \partial_y \rangle = 0.$$

The purpose of this problem is to show that the fractional linear transformations $F$ are isometries of the Poincaré metric. This can be done by direct computation but it is more efficient to do it in simpler steps.

a) Prove that $F(i) = i$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2, \mathbb{R})$$

and that in this case,

$$F_{*,i} : T_i(\mathcal{H}) \to T_i(\mathcal{H}).$$

is an isometry of the Poincaré metric at $i \in \mathcal{H}$.

b) Let $a > 0$ and set

$$S_a = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

Prove that $S_a$ is an isometry of $(\mathcal{H}, \langle , \rangle)$ for all $a > 0$.

c) Prove that every transformation $F$ as in (1) may be written as

$$F = S_a \circ W ; \quad a > 0, \quad W(i) = i$$
d) Let $F$ be as in (1). Prove that
\[ F_{*i} : T_i(\mathcal{H}) \rightarrow T_{F(i)}(\mathcal{H}) \]
is an isometry with respect to the Poincaré metric.
e) Prove that every $F$ as in (1) is an isometry of $\mathcal{H}$.

*Problem 30. Find a local coordinate expression for the gradient $\text{grad}(f)$ in the following cases:

a) The upper-half plane $\mathcal{H}$ with the Poincaré metric and the usual coordinates $(x, y)$.
b) The sphere $S^2$ with the Euclidean metric and the coordinates defined by stereographic projection.
c) The Heisenberg group $H_3$ with a left invariant metric of your choosing and the usual coordinates $(x, y, z)$.

*Problem 31. Let $(M, g)$ be a Riemannian manifold and $\lambda \in C^\infty(M)$. We define a new Riemannian metric on $M$ by
\[ g_\lambda(p)(u, v) := e^{\lambda} g(p)(u, v) ; \quad u, v \in T_p(M). \]

a) Verify that $g_\lambda$ is a Riemannian metric on $M$.
b) Given $f \in C^\infty(M)$, is $\text{grad}_g(f) = \text{grad}_{g_\lambda}(f)$? If not, how are they related.

*Problem 32. Let $F : \mathbb{P}^2 \subset \mathbb{R}^4$ be an embedding (see for example, Problem 23 from M703). Let $X = F(\mathbb{P}^2)$. Prove that there is no map $G : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ such that $0$ is a regular value for $G$ and $X = F^{-1}(0)$.

*Problem 33. Let $\pi_i : E_i \rightarrow M, \quad i = 1, 2$ be vector bundles and suppose $E_2$ is a subbundle of $E_1$ and that $E_1$ is orientable. Prove that $E_2$ is orientable if and only if the quotient $E_1/E_2$ is orientable.

*Problem 34. Let $M \subset \mathbb{R}^n$ be an $(n - 1)$-dimensional submanifold. Show that $M$ is orientable if and only if $\nu(M, \mathbb{R}^n)$ is trivial.