Only hand-in problems preceded by a *.

**Problem 13.** Prove that the operation
\[(x_1, y_1) \ast (x_2, y_2) := (x_1x_2, y_1 + x_1y_2)\]
turns the open set
\[G = \{(x, y) \in \mathbb{R}^2 : x > 0\}\]
into a Lie group. Find its Lie algebra \(g\) and describe the exponential map:
\[\exp : g \to G.\]

**Problem 14.** Let \(G \subset GL(n, \mathbb{R})\) be a Lie subgroup and \(g \subset gl(n, \mathbb{R})\) its Lie algebra. For \(X \in G\), let
\[\varphi_X : G \to G\]
be the automorphism \(\varphi_X(Y) = X \cdot Y \cdot X^{-1}\). and
\[\text{Ad}(X) := (\varphi_X)_*: g \to g\]
its differential.

a) Compute \(\text{Ad}(X)\) and prove that \(\text{Ad}(X)\) is invertible; i.e. \(\text{Ad}(X) \in GL(g)\).

b) Prove that the map \(\text{Ad} : G \to GL(g), X \mapsto \text{Ad}(X)\) is a \(C^\infty\) map.

c) Prove that the map
\[\text{ad} := \text{Ad}_*: g \to gl(g)\]
is given by
\[(\text{ad}(A))(B) = [A, B] ; \quad A, B \in g.\]

**Problem 15.** Consider the map
\[\varphi : SO(n, \mathbb{R}) \to S^{n-1}\]
defined as \(\varphi(X) = X \cdot e_1\), where \(e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^n\).

a) Prove that \(\varphi\) is \(C^\infty\).

b) Prove that \(\varphi\) is a submersion.

c) Describe the manifolds \(M_v := \varphi^{-1}(v), v \in S^{n-1}\).

d) Suppose \(n = 3\). In the lecture notes we described a two-fold covering map \(S^3 \to SO(3, \mathbb{R})\). Composing with \(\varphi\) we get a map
\[H : S^3 \to S^2\]
Write \(H\) explicitly. Prove that \(H\) is a submersion and that for every \(v \in S^2\), \(H^{-1}(v) \cong S^1\). (Note: \(H\) is called the Hopf fibration.)
Problem 16. Let $\mathfrak{g}$ be a Lie algebra (not necessarily a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$) and let

$$\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

be the map

$$(\text{ad}(A))(B) = [A, B] ; \ A, B \in \mathfrak{g}.$$ 

We say that $\mathfrak{g}$ is unimodular if and only if for every $A \in \mathfrak{g}$ the linear map $\text{ad}(A) : \mathfrak{g} \to \mathfrak{g}$ satisfies

$$\text{tr} (\text{ad}(A)) = 0 .$$

The purpose of this problem is to classify all unimodular three-dimensional Lie algebras. We assume that then that the underlying vector space of $\mathfrak{g}$ is $\mathbb{R}^3$. Let $E^3$ denote Euclidean 3-space; i.e. $\mathbb{R}^3$ with the Euclidean inner product and usual orientation. Let $\times$ denote the cross product in $E^3$.

a) Prove that $(E^3, \times)$ is a unimodular Lie algebra.

b) Prove that if $(\mathbb{R}^3, [ , ])$ is a Lie algebra then there exists a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$[u, v] = L(u \times v).$$

c) Prove that if $(\mathbb{R}^3, [ , ])$ is unimodular then the matrix of $L$ in the standard basis is symmetric. Hence $L$ may be diagonalized over $\mathbb{R}$.

d) Prove that if $L_1 = D \cdot L_2$, where $D$ is a diagonal matrix with positive diagonal entries then the corresponding Lie algebras are isomorphic.

e) Show that there are six isomorphism classes of unimodular three dimensional Lie algebras.