Problem 9. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R}, \ f(x, y) = x^3 + xy + y^3 + 1. \)

a) Find the differential \( Df_p, \) for \( p = (a, b) \in \mathbb{R}^2. \)

b) For which points \( p = (a, b) \in \mathbb{R}^2 \) is \( Df_p \) injective? onto?

Problem 10. Let \( H : \mathbb{R}^3 \to \mathbb{R}^2 \) be the map:
\[
H(x, y, z) = (xy, x^2z).
\]
Find all points \( p \in \mathbb{R}^3 \) at which the differential \( DH_p \) is not onto.

*Problem 11. Let \( F : \mathbb{R}^3 \to \mathbb{R}^5 \) be the map:
\[
F(x, y, z) = (x^2, xy, xz + y^2, yz, z^2).
\]
Find all points \( p \in \mathbb{R}^3 \) at which the differential \( DF_p \) has rank 3, 2, 1, or 0.

*Problem 12. Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map:
\[
F(x, y) = (x, y - y^2 - x^2)
\]

a) Prove that \( F \) is a local diffeomorphism at the origin. Compute a local inverse for \( F \) at the origin.

b) Exhibit a point \( p \in \mathbb{R}^2 \) such that \( F \) is not a local diffeomorphism at \( p. \)

*Problem 13. Give a direct proof of the following special case of the Rank Theorem: Let \( n > m, \) \( A \) an open subset of \( \mathbb{R}^n. \) Suppose \( F : A \to \mathbb{R}^m \) is a \( C^\infty \) map and that for some \( p \in A, \) \( \text{rank}_p F = m. \) Then, there exist open neighborhoods \( A_1 \subset A \) of \( p, \) and \( B_1 \subset \mathbb{R}^m \) of \( F(p), \) and diffeomorphisms \( \varphi : A_1 \to U, \psi : B_1 \to V \) such that \( \psi(F(\varphi^{-1}(U))) \subset V \) and
\[
\psi(F(\varphi^{-1}(x_1, \ldots, x_n))) = (x_1, \ldots, x_m)
\]

Problem 14. Let \( I : \mathbb{R}^{n+1} \setminus \{0\} \to S^n \) be defined by
\[
I(x) = \frac{x}{\|x\|}
\]
Prove that \( I \) is a submersion.

*Problem 15. Let \( GL(n, \mathbb{R}) \) denote the space of \( n \times n \) invertible matrices with real coefficients. We view \( GL(n, \mathbb{R}) \) as a subset of the space of all \( n \times n \) matrices which we identify with \( \mathbb{R}^{n^2}. \)
a) Prove that $GL(n, \mathbb{R})$ is open in $\mathbb{R}^{n^2}$.

b) Prove that $\sigma: GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $\sigma(M) = M^2$ is a $C^\infty$ map.

c) Prove that there exists open neighborhoods $U, V$ of the identity matrix, such that for every $X \in V$ there exists $S \in U$ such that $S^2 = X$. That is, we can find square roots of matrices in a neighborhood of the identity.

**Problem 16.** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$ function such that

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^a f(x_1, \ldots, x_n)$$

for all $\lambda \in \mathbb{R}$ and some $a \in \mathbb{R}$. ($f$ is said to be homogeneous of degree $a$). Prove that

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = af.$$

**Problem 17.** Let $M$ and $N$ be $C^\infty$-manifolds. Suppose that $\{(U_\alpha, \phi_\alpha); \alpha \in A\}$ and $\{(V_\beta, \psi_\beta); \beta \in B\}$ are atlases in $M$ and $N$, respectively.

a) Show that $M \times N$ is a topological manifold (with the product topology) and that

$$\{(U_\alpha \times V_\beta, (\phi_\alpha, \psi_\beta)); (\alpha, \beta) \in A \times B\}$$

is an atlas for $M \times N$.

b) Let $(p, q) \in M \times N$. Define maps

$$\iota_q: M \to M \times N ; \quad \iota_q(x) := (x, q)$$

and, similarly, $\iota_p: N \to M \times N$. Prove that $\iota_p$ and $\iota_q$ are embeddings.

c) Let $Z$ be a $C^\infty$-manifold and suppose $F: Z \to M$ and $G: Z \to N$ are $C^\infty$ maps. Prove that

$$F \times G: Z \to M \times N ; \quad F \times G(z) := (F(z), G(z))$$

is a $C^\infty$ map.

*Problem 18.** Let $F: M \to N$ be a submersion. Prove that $F$ is an open map.

*Problem 19.** Let $M, N, X$ be $C^\infty$ manifolds and suppose $N \subset M$. Let $F: X \to M$ be a $C^\infty$ map such that $F(X) \subset N$.

a) Show that if $N$ is an immersed submanifold of $M$ and $F: X \to N$ is continuous, then $F: X \to N$ is a $C^\infty$ map.

b) Show that if $N$ is an embedded submanifold of $M$ then $F: X \to N$ is a $C^\infty$ map.

c) Give an example to show that if $N \subset M$ is an (immersed) submanifold, there may be a $C^\infty$ function $f: N \to \mathbb{R}$ which is not the restriction of a $C^\infty$ function $g: M \to \mathbb{R}$. What happens if $N$ is an embedded submanifold?