11) a) Problem 17, page 29. Alternatively, you may prove the following strongest result: Let \( \{a_n\} \) be a sequence of positive real numbers, then:

\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \to \infty} a_n^{1/n} \leq \limsup_{n \to \infty} a_n^{1/n} \leq \limsup_{n \to \infty} \frac{a_{n+1}}{a_n}
\]

b) Use part (a) to prove the following weak version of Stirling’s formula:

\[
\lim_{n \to \infty} \frac{n}{(n!)^{1/n}} = e.
\]

12) a) Problem 16, page 28. Replace the use of Stirling’s formula by the weaker version of 11 b).

13) Problems 19 (a) and (b), page 29.


16) Suppose \( S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \) converges absolutely for \( |z - z_0| < R \) and suppose \( S(z) \) is constant for \( |z - z_0| < \epsilon < R \), for some \( \epsilon > 0 \). Show that \( a_n = 0 \) for \( n \geq 1 \).

For the following two problems we take as definition of \( e^z \) the one given in class, that is:

\[
e^z := e^x \cos y + i \sin y.
\]

We have shown in class that this is an entire function.

17) Suppose \( S(z) = \sum_{n=0}^{\infty} a_n z^n \) converges absolutely for \( |z| < R \).

a) Show that

\[
\text{Re}(S(x + iy)) = \sum_{n=0}^{\infty} \text{Re}(a_n z^n)
\]

b) Let \( S(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \). Prove that \( \text{Re}(S(x+iy)) = e^x \cos y \) and deduce that \( S(z) = e^z \). (In particular, you may not assume that \( S(z) = e^z \).)
18) Let \( S(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \) and \( g(z) = e^z = e^x(\cos y + i \sin y) \). (The purpose of this problem is to give yet another proof that \( S(z) = e^z \), so do not assume this fact)

a) Show that \( h(z) = \frac{S(z)}{g(z)} \) is an entire function.

b) Show that \( h'(z) \equiv 0 \) and deduce that \( S(z) = e^z \).

19) Let \( \gamma : [a, b] \to \mathbb{C} \) be a \( C^1 \) curve. Suppose \( \{f_n\} \) is a sequence of \( \mathbb{C} \)-valued functions, defined and continuous in an open set \( W \subset \mathbb{C} \) containing \( \gamma([a, b]) \). Prove that if \( f_n \) converges, uniformly on \( \gamma([a, b]) \) to a function \( f \), then

\[
\lim_{n \to \infty} \int_{\gamma} f_n(z) \, dz = \int_{\gamma} f(z) \, dz
\]

20) Compute the following line integrals:

a) \( \int_{\gamma} (\bar{z} + z^2) \, dz \), where \( \gamma \) is the square with vertices at the points: \( \pm 1 \pm i \).

b) \( \int_{\gamma} (z^2 + 2z + 3) \, dz \), where \( \gamma \) is the segment joining 1 to 2 + i.

c) \( \int_{\gamma} \frac{\bar{z}}{8 + z} \, dz \), where \( \gamma \) is the rectangle with vertices at the points: \( \pm 3 \pm i \).