INTRODUCTION TO VARIATIONS OF HODGE STRUCTURE SUMMER SCHOOL ON HODGE THEORY, ICTP, JUNE 2010

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Preliminary Version

Introduction

These notes are intended to accompany the course Introduction to Variations of Hodge Structure (VHS) at the 2010 ICTP Summer School on Hodge Theory.

The modern theory of variations of Hodge structure (although some authors have referred to this period as the pre-history) begins with the work of Griffiths [23, 24, 25] and continues with that of Deligne [17, 18, 19], and Schmid [41]. The basic object of study are *period domains* which parametrize the possible polarized Hodge structures in a given smooth projective variety. An analytic family of such varieties gives rise to a holomorphic map with values in a period domain, satisfying an additional system of differential equations. Moreover, period domains are homogeneous quasi-projective varieties and, following Griffiths and Schmid, one can apply Lie theoretic techniques to study these maps.

These notes are not intended as a comprehensive survey of the theory of VHS. We refer the reader to the surveys [25, 30, 2, 1, 33], the collections [26, 1], and the monographs [4, 40, 45, 46] for fuller accounts of various aspects of the theory. In these notes we will emphasize the theory of *abstract* variations of Hodge structure and, in particular, their asymptotic behavior. The geometric aspects will be the subject of the subsequent course by James Carlson.

In §1, we study the basic correspondence between local systems, representations of the fundamental group, and bundles with a flat connection. We also define the Kodaira-Spencer map associated with a family of smooth projective varieties. The second section is devoted to the study of Griffiths' period map and a discussion of its main properties: holomorphicity and horizontality. These properties motivate the notion of an abstract VHS. In §3, we define the classifying spaces for polarized Hodge structures and study some of their basic properties. The last two sections deal with the asymptotics of a period mapping with particular attention to Schmid's Orbit Theorems. We emphasize throughout this discussion the relationship between nilpotent and SL_2-orbits and mixed Hodge structures.

In these notes I have often drawn from previous work in collaboration with Aroldo Kaplan, Wilfried Schmid, Pierre Deligne, and Javier Fernandez to all of whom I am very grateful.

A final version of these notes will be posted after the conclusion of the Summer School.

Contents

1. Analytic families

1.1. **The Kodaira-Spencer Map.** We will be interested in considering families of compact Kähler manifolds or smooth projective varieties varying holomorphically on a base of parameters. Specifically, consider a map

$$\varphi\colon \mathcal{X} \to B, \tag{1.1}$$

where $\mathcal{X}$ and B are complex manifolds, and φ is a proper, holomorphic submersion; i.e. φ is surjective and, for every $x \in \mathcal{X}$, the differential

$$\varphi_{*,x}\colon T_x(\mathcal{X}) \to T_{\varphi(x)}(B)$$

is also surjective.

It follows from [7, Theorem 1.9] that for each $b \in B$, the fiber $X_b := \varphi^{-1}(b)$ is a complex submanifold of $\mathcal{X}$ of codimension equal to the dimension of B. Moreover, since φ is proper, X_b is compact. We think of $\{X_b;\ b \in B\}$ as an analytic family of compact complex manifolds. The following theorem asserts that, $\varphi\colon \mathcal{X} \to B$ is a C^∞ fiber bundle; i.e. it is locally a product:

Theorem 1.1. *For every $b_0 \in B$ there exists a polydisk $\mathcal{U}$ centered at b_0 and a C^∞ map $F\colon \varphi^{-1}(\mathcal{U}) \subset \mathcal{X} \to \mathcal{U} \times X_{b_0}$ such that the diagram*

$$\begin{tikzcd} \varphi^{-1}(\mathcal{U}) \arrow[rr, "F"] \arrow[dr, "\varphi"'] & & \mathcal{U} \times X_{b_0} \arrow[dl, "\mathrm{pr}_1"] \\ & \mathcal{U} & \end{tikzcd} \tag{1.2}$$

where pr_1 is the projection on the first factor, commutes. Moreover, for every $x \in X_{b_0}$ the map

$$\sigma_x\colon \mathcal{U} \to \mathcal{X}\ ;\quad \sigma_x(b) \ :=\ F^{-1}(b,x), \tag{1.3}$$

is holomorphic.

Remark 1. For the first statement to hold, it suffices to assume that $\mathcal{X}, B$ and φ are smooth. In that context it is a well-known result due to Ehresmann. In fact, the family φ trivializes over any contractible neighborhood of $b \in B$. We refer to [45, Theorem 9.3] for a complete proof of both statements.

In what follows, we will assume that $B = \mathcal{U}$ and we have chosen local coordinates $(t_1, \ldots, t_r)$ in B centered at b_0. We set $X = X_{b_0} = X_0$. Let G denote the inverse of the diffeomorphism $F\colon \mathcal{X} \to B \times X$. Even though for every $t \in B$, the fibre X_t is a complex submanifold of $\mathcal{X}$, the restriction

$$g_t := G|_{\{t\}\times X}\colon X \to X_t\ ,\quad t \in B$$

is, generally, only a diffeomorphism and carries the complex structure J_{X_t} to a $(1,1)$ tensor $J_t := g_t^*(J_{X_t})$ on X satisfying $J_t^2 = -\,\mathrm{id}$. Moreover, it follows from [7, Theorem 2.2] that J_t is integrable. Thus, we may, alternatively, think of $\{X_t\}$ as a family of complex structures on a fixed C^∞ manifold X.

Let TX_0, $T\mathcal{X}$, and TB denote the tangent bundles of X, $\mathcal{X}$ and B, respectively. Recalling that for each $x \in X \subset \mathcal{X}$,

$$T_x(X) \ =\ \ker\{\varphi_{*,x}\colon T_x\mathcal{X} \to T_0B\}$$

we have an exact sequence of vector bundles over X:

$$0 \to TX \hookrightarrow T\mathcal{X}|_X \stackrel{\varphi_*}{\to} X \times T_0(B) \to 0. \tag{1.4}$$

On the other hand, the fact that φ is a submersion means that we also have an exact sequence of bundles over $\mathcal{X}$:

$$0 \to T_{\mathcal{X}/B} \to T\mathcal{X} \stackrel{\varphi_*}{\to} \varphi^*(TB) \to 0, \tag{1.5}$$

where $\varphi^*(TB)$ is the pull-back bundle defined in [7, (1.19)] and the *relative* bundle $T_{\mathcal{X}/B}$ is defined as the kernel of φ_*.

Since φ is holomorphic these maps are compatible with the complex structures and, therefore, we get analogous exact sequences of holomorphic tangent bundles.

$$0 \to T^hX \hookrightarrow T^h\mathcal{X}|_X \stackrel{\varphi_*}{\to} X \times T_0^h(B) \to 0. \tag{1.6}$$

$$0 \to T^h_{\mathcal{X}/B} \to T^h\mathcal{X} \stackrel{\varphi_*}{\to} \varphi^*(T^hB) \to 0, \tag{1.7}$$

The sequence (1.6) gives rise to an exact sequence of sheaves of holomorphic sections and, consequently, to a long exact sequence in cohomology yielding, in particular, a map:

$$H^0(X, \mathcal{O}(X \times T_0^h(B))) \to H^1(X, \mathcal{O}(T^hX)), \tag{1.8}$$

where $\mathcal{O}(T^h(X))$ is the sheaf of holomorphic vector fields on X.

Since X is compact, any global holomorphic function is constant and, consequently $H^0(X, \mathcal{O}(X \times T_0^h(B))) \cong T_0^h(B)$. On the other hand, it follows from the Dolbeault isomorphism theorem that:

$$H^1(X, \mathcal{O}(T^hX)) \cong H^{0,1}_{\bar\partial}(X, T^hX). \tag{1.9}$$

Definition 1.2. The map

$$\rho \colon T_0^h(B) \to H^1(X, \mathcal{O}(T^hX)) \cong H^{0,1}_{\bar\partial}(X, T^hX) \tag{1.10}$$

is called the *Kodaira-Spencer* map at $t = 0$.

We may obtain a description of ρ using the map σ_x defined in (1.3). Indeed, for each $v \in T_0^h(B)$, let us denote by V the constant holomorphic vector field on B whose value at 0 is v. We may regard V as a holomorphic vector field on $B \times X$ and we define a C^∞ vector field Y_v on $\mathcal{X}$ by $Y_v = G_*(V)$. Note that for $x \in X$, $t \in B$,

$$Y_v(\sigma_x(t)) \;=\; (\sigma_x)_{*,t}(V) \tag{1.11}$$

and, therefore, Y_v is a vector field of type $(1,0)$. Moreover,

$$\phi_*(Y_v(\sigma_x(t))) \;=\; V(t),$$

and, therefore, Y_v is the unique smooth vector field of type $(1,0)$ on $\mathcal{X}$ projecting to V. In local coordinates $(U, \{z_1^U, \ldots, z_n^U\})$ the vector field $Y_v|_X$ may be written as

$$Y_v(x) \;=\; \sum_{j=1}^n \nu_j^U \frac{\partial}{\partial z_j^U}. \tag{1.12}$$

Since the coordinate changes in T^hX:

$$\left(\frac{\partial z_k^V}{\partial z_j^U}\right)$$

are holomorphic one can show:

Exercise 1. The expression

$$\alpha_v \;=\; \sum_{j=1}^{n} \bar\partial(\nu_j^U) \otimes \frac{\partial}{\partial z_j^U} \tag{1.13}$$

defines a global $(0,1)$ form on X with values on the holomorphic tangent bundle T^hX.

Following the steps involved in the proof of the Dolbeault isomorphism it is easy to check that $[\alpha_v]$ is the cohomology class in $H^{0,1}_{\bar\partial}(X, T^hX)$ corresponding to $\rho(v)$ in (1.10).

We will give a different description of the form α_v which motivates the definition of the Kodaira-Spencer map: As noted above, the family $\varphi\colon \mathcal{X} \to B$ gives rise to a family $\{J_t : t \in B\}$ of almost complex structures on X. As we saw in [7, Proposition A.1] such an almost complex structure is equivalent to a splitting for each $x \in X$:

$$T_{x,\mathbb{C}}(X) \;= (T_x)_t^+ \oplus (T_x)_t^- \;;\quad (T_x)_t^- = \overline{(T_x)_t^+},$$

and where $(T_x)_0^+ = T_x^h(X)$. If t is small enough we may assume that the projection of $(T_x)_t^-$ on $(T_x)_0^-$, according to the decomposition corresponding to $t = 0$, is surjective. Hence, in a coordinate neighborhood $(U, \{z_1^U, \ldots, z_n^U\})$, there is a basis[†] of the subspace $(T_x)_t^-$, $x \in U$, of the form

$$\frac{\partial}{\partial \bar z_k^U} - \sum_{j=1}^{n} w_{jk}^U(z,t)\,\frac{\partial}{\partial z_j^U} \;;\quad k = 1, \ldots, n.$$

Thus, the local expression

$$\sum_{j=1}^{n} w_{jk}^U(z,t)\,\frac{\partial}{\partial z_j^U}$$

describes, in local coordinates, how the almost complex structure varies with $t \in B$. Now, given $v \in T_0^h(B)$ we can "differentiate" the above expression with respect to t in the direction of v to get [‡]:

$$\sum_{j=1}^{n} v\left(w_{jk}^U(z,t)\right)\,\frac{\partial}{\partial z_j^U}.$$

We then have the following result whose proof may be found in [45, §9.1.2]:

[†]Note that this is essentially the notation in Example 1.16 of [7].

[‡]Note that in what follows we could as well assume $v \in T_0(B)$; i.e. v need not be of type $(1,0)$.

Proposition 1.3. *For each $v \in T^h_0(B)$ the expression*

$$\sum_{j=1}^{n} \bar\partial\left(v\left(w_{jk}(z,t)\right)\right) \otimes \frac{\partial}{\partial z_j^U}.$$

defines a global $(0,1)$ *form on* X *with values on the holomorphic tangent bundle* T^hX*, whose cohomology class in* $H^{0,1}_{\bar\partial}(X, T^hX)$ *agrees with the Kodaira-Spencer class* $\rho(v)$*.*

Our next goal is to globalize these constructions to a non-contractible base B.

1.2. **Local Systems.** In the next two sections we will collect some basic results about local systems and bundles with flat connections. We refer to [16] and [45, §9.2] for details.

We recall that the constant sheaf with stalk $\mathbb{C}^n$ is the sheaf of $\mathbb{C}$-vectorspaces whose sections over any open set U is the vector space $\mathbb{C}^n$.

Definition 1.4. A sheaf $\mathcal{L}$ over B is called a local system of $\mathbb{C}$-vectorspaces if it is locally isomorphic to a constant sheaf with stalk $\mathbb{C}^n$ for a fixed n.

If $\mathcal{L} \to B$ is a local system, and we fix a base point $b_0 \in B$, then for any curve $\gamma\colon [0,1] \to B$, $\gamma(0) = b_0$, $\gamma(1) = b_1$, the pull-back $\gamma^*(\mathcal{L})$ to $[0,1]$ is locally constant, hence constant. Thus we get an $\mathbb{C}$-vectorspace isomorphism:

$$\tau^\gamma\colon \mathcal{L}_{b_1} \to \mathcal{L}_{b_0},$$

which depends only on the homotopy class of the path γ. Taking closed loops based at b_0, we get a map:

$$\rho\colon \pi_1(B, b_0) \to \mathrm{GL}(\mathcal{L}_{b_0}) \cong \mathrm{GL}(n, \mathbb{C}). \tag{1.14}$$

It is easy to check that ρ is a group homomorphism and, consequently, it defines a representation of the fundamental group $\pi_1(B, b_0)$ on $\mathcal{L}_{b_0} \cong \mathbb{C}^n$. If B is connected, this construction is independent, up to conjugation, of the base point b_0. We will assume throughout that B is *connected*.

Conversely, suppose $\rho\colon \pi_1(B, b_0) \to \mathrm{GL}(n, \mathbb{C})$ is a finite-dimensional representation and let $\mathfrak{p}\colon \tilde B \to B$ be the universal covering space of B. The fundamental group $\pi_1(B, b_0)$ acts on $\tilde B$ by covering (deck) transformations† and we may define a holomorphic vector bundle $\mathbb{V} \to B$ by

$$\mathbb{V} := \tilde B \times \mathbb{C}^n / \sim\,, \tag{1.15}$$

where the equivalence relation $\sim$ is defined as

$$(\tilde b, v) \sim (\sigma(\tilde b), \rho(\sigma^{-1})(v))\ ; \quad \sigma \in \pi_1(B, b_0), \tag{1.16}$$

and the map $\mathbb{V} \to B$ is the natural projection from $\tilde B$ to B.‡ Suppose $U \subset B$ is an evenly covered open set in B, that is $\mathfrak{p}^{-1}(U)$ is a disjoint union of open sets $W_j \subset \tilde B$

†Since the group action on $\pi_1(B, b_0)$ is defined by concatenation of loops, the action on the universal covering space is a right action.

‡In other words, $\mathbb{V}$ is the vector bundle associated to the principal bundle $\pi_1(B, b_0) \to \tilde B \to B$ by the representation ρ

biholomorphic to U. Let us denote by $\mathfrak{p}_j = \mathfrak{p}|_{W_j}$. Then, given any $v \in \mathbb{C}^n$ we have, for any choice of j, a local section

$$\hat{v}(z) = [\mathfrak{p}_j^{-1}(z), v] \ ; \quad z \in U$$

on U. We call $\hat{v}$ a *constant section* of the bundle $\mathbb{V}$ and note that this notion is well defined since the transition functions of the bundle $\mathbb{V}$ take value on the discrete group $\rho(\pi_1(B, b_0))$. We denote by $\mathcal{L}$ the sheaf of constant local sections of $\mathbb{V}$. Clearly, $\mathcal{L}$ is a locally constant sheaf, i.e. a local system. As we shall see below, every local system arises in this way from a certain class of holomorphic vector bundles.

Example 1.5. Let $B = \Delta^* := \{z \in \mathbb{C} : 0 < |z| < r\}$, where we assume, for simplicity that we have scaled our variable so that $r > 1$. For $t_0 = 1 \in \Delta^*$ we have $\pi_1(\Delta^*, t_0) \cong \mathbb{Z}$, where we choose as generator a simple loop c oriented clockwise. Let

$$\rho\colon \pi_1(\Delta^*, t_0) \cong \mathbb{Z} \to \mathbb{C}^2 \ ; \quad \rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}). \tag{1.17}$$

Recalling that the upper half-plane $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ is the universal covering space of Δ^* with projection $z \mapsto \exp(2\pi i z)$, we have a commutative diagram:

$$\begin{array}{ccc} H \times \mathbb{C}^2 & \longrightarrow & \mathbb{V} \cong H \times \mathbb{C}^2/\sim \\ {\scriptstyle pr_1}\big\downarrow & & \big\downarrow \\ H & \xrightarrow{\exp(2\pi i \bullet)} & \Delta^* \end{array}$$

Let N be the nilpotent transformation

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.18}$$

Then, for any $v \in \mathbb{C}^2$, the map $\tilde{v}\colon \Delta^* \to \mathbb{V}$ defined by

$$\tilde{v}(t) := \left[\frac{\log t}{2\pi i} \ , \ \exp\left(\frac{\log t}{2\pi i} N \right) \cdot v \right] \in H \times \mathbb{C}^2/\sim \tag{1.19}$$

is a section of the vector bundle $\mathbb{V}$. Indeed, suppose we follow a determination of log around the loop c, then $z_0 = \log(t_0/2\pi i)$ changes to $z_0 - 1$ (i.e. $\rho(c)(z_0) = z_0 - 1$), while the second component is modified by the linear transformation $\exp(-N) = \rho(c^{-1})$ as required by (1.16). Note that, on a contractible neighborhood U of $t \in \Delta^*$, we can write

$$\tilde{v}(t) = \exp((\log t/2\pi i)N) \cdot \hat{v}(t),$$

where $\hat{v}(t)$ is the constant section defined on U.

This example may be generalized to an arbitrary nilpotent transformation $N \in \mathfrak{gl}(V)$ of a $\mathbb{C}$-vectorspace V if we define

$$\rho\colon \pi_1(\Delta^*, t_0) \to \mathrm{GL}(V)$$

by $\rho(c) = \exp(N)$, where c is, again, a simple loop oriented clockwise, and to commuting nilpotent transformations $\{N_1, \ldots, N_r\} \in \mathfrak{gl}(V)$ by considering $B = (\Delta^*)^r$ and

$$\rho\colon \pi_1((\Delta^*)^r, t_0) \cong \mathbb{Z}^r \to \mathrm{GL}(V)$$

the representation that maps the j-th standard generator of $\mathbb{Z}^r$ to $\gamma_j = \exp N_j$.

1.3. **Flat Bundles.** As we saw in the previous section, a local system over B gives rise to a representation of the fundamental group of B which, in turn, may be used to construct a vector bundle with a subspace of "distinguished" constant sections isomorphic to the original local system. Here, we want to explore what is involved in the existence of this subspace of constant sections from the point of view of the bundle itself. We refer to [16, 34] for details.

Recall that a holomorphic connection on a holomorphic vector bundle $E \to B$ is a $\mathbb{C}$-linear map:

$$\nabla\colon \mathcal{O}(U,E) \to \Omega^1(U) \otimes \mathcal{O}(U,E) := \Omega^1(U,E), \tag{1.20}$$

where $U \subset B$ is an open set and such that

$$\nabla(f\cdot\sigma) \;=\; df\otimes\sigma + f\cdot\nabla\sigma\ ;\quad f\in\mathcal{O}(U),\ \sigma\in\mathcal{O}(U,E). \tag{1.21}$$

In terms of a local holomorphic coframe $\sigma_1,\ldots,\sigma_d$ of $\mathcal{O}(U,E)$, we can write:

$$\nabla\sigma_j \;=\; \sum_{i=1}^d \theta_{ij}\otimes\sigma_i. \tag{1.22}$$

The holomorphic forms $\theta_{ij}\in\Omega^1(U)$ are called *connection forms*.

Definition 1.6. Let $E\colon B$ be a bundle with a connection ∇. A section $\sigma\in\mathcal{O}(U,E)$ is said to be *flat* if $\nabla\sigma = 0$. The connection ∇ is called *flat* if there is a trivializing cover of B for which the corresponding coframe consists of flat sections.

A connection on a holomorphic line bundle $E\to B$ allows us to differentiate holomorphic (resp. smooth) sections of E in the direction of a holomorphic vector field X on $U\subset B$. Indeed, for U small enough and a coframe $\sigma_1,\ldots,\sigma_d$ of $\mathcal{O}(U,E)$ we set:

$$\nabla_X\left(\sum_{j=1}^d f_j\sigma_j\right) \;:=\; \sum_{i=1}^d \left(X(f_i) + \sum_{j=1}^d f_j\,\theta_{ij}(X)\right)\sigma_i\,.$$

Clearly, if the coefficients f_j are holomorphic, so is the resulting section.

Exercise 2. Prove that the connection forms must satisfy the following *compatibility condition*: If $\sigma'_1,\ldots,\sigma'_d$ is another coframe on U and

$$\sigma'_j = \sum_{i=1}^d g_{ij}\sigma_i\ ;\quad g_{ij}\in\mathcal{O}(U),$$

then

$$\sum_{i=1}^d g_{ji}\,\theta'_{ik} = dg_{jk} + \sum_{i=1}^d \theta_{ji}\,g_{ik}. \tag{1.23}$$

Deduce that if we define the matrices: $\theta = (\theta_{ij})$, $\theta' = (\theta'_{ij})$, $g = (g_{ij})$, $dg = (dg_{ij})$, then

$$\theta' \;=\; g^{-1}\cdot dg + g^{-1}\cdot\theta\cdot g. \tag{1.24}$$

Exercise 3. Let $L \to M$ be a line bundle and suppose that U_α is a trivializing cover of M with transition functions $g_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$. Prove that a connection on M is given by a collection of holomorphic one-forms $\theta_\alpha \in \Omega^1(U_\alpha)$ such that:

$$\theta_\beta|_{U_\alpha \cap U_\beta} - \theta_\alpha|_{U_\alpha \cap U_\beta} = d(\log g_{\alpha\beta}). \tag{1.25}$$

The curvature matrix of a connection ∇ is defined as the matrix of holomorphic two-forms:

$$\Theta_{ij} = d\theta_{ij} - \sum_{k=1}^{d} \theta_{ik} \wedge \theta_{kj}, \tag{1.26}$$

or, in matrix notation:

$$\Theta = d\theta - \theta \wedge \theta.$$

Exercise 4. With the notation of Exercise 2, prove that

$$\Theta' = g^{-1} \cdot \Theta \cdot g. \tag{1.27}$$

The curvature forms measure the "failure" of the connection to be flat:

Theorem 1.7. *A connection is flat if and only if the curvature forms are identically zero.*

Proof. We note, first of all, that (1.27) implies that the vanishing of the curvature forms is independent of the choice of coframe. On the other hand, if ∇ is flat, we can find a trivializing cover where the connection forms and, therefore, the curvature forms vanish.

Suppose $(U, z_1, \ldots, z_n)$ is a coordinate neighborhood on B such that there exists a local coframe $\sigma_1, \ldots, \sigma_d$ of $\mathcal{O}(U, E)$. This allows us to define coordinates $\{z_1, \ldots, z_n, \xi_1, \ldots, \xi_d\}$ on E. The forms

$$d\xi_i + \sum_{j=1}^{d} \xi_i \, \theta_{ij}$$

define a distribution of dimension n on E corresponding to "flat liftings". The existence of an n-dimensional integral manifold is equivalent to the existence of a flat local coframe. The distribution is involutive if and only if the curvature forms vanish. Thus, the result follows from Frobenius Theorem. We refer to [34, Proposition 2.5], [45, §9.2.1] for a full proof. □

Suppose now that a vector bundle $\mathbb{V} \to B$ arises from a local system $\mathcal{L}$ as before. Then, the bundle V has a trivializing cover relative to which the transition functions are constant (since they take values in a discrete subgroup of $\mathrm{GL}(n, \mathbb{C})$), and it follows from (1.23) that the local forms $\theta_{ij} = 0$ define a connection on $\mathbb{V}$; that is, relative to the coframe $\sigma_1^\alpha, \ldots, \sigma_d^\alpha$ arising from that trivializing cover, we may define

$$\nabla\left(\sum_{i=1}^{d} f_i \, \sigma_i^\alpha\right) = \sum_{i=1}^{d} df_i \otimes \sigma_i^\alpha. \tag{1.28}$$

Since the curvature forms for ∇ vanish, it follows that ∇ is flat.

Conversely, suppose $E \to B$ is a bundle with a flat connection ∇. Then the transition functions corresponding to the covering by open sets with flat coframes must be constant. Consequently we can define a local system of constant sections, i.e. the flat sections.

Summarizing the results of these two sections we can say that there is an equivalence between the following three categories:

i) Local systems over a connected, complex manifold B.
ii) Finite-dimensional representations of the fundamental group $\pi_1(B, b_0)$.
iii) Holomorphic bundles $\mathbb{V} \to B$ with a flat connection ∇.

1.4. **The Gauss-Manin Connection.** We return now to the case of an analytic family $\varphi\colon \mathcal{X} \to B$ of compact, complex manifolds parametrized by the complex manifold B. By Theorem 1.1, $\mathcal{X}$ is locally trivial as a C^∞ manifold. We write as before,

$$\varphi^{-1}(\mathcal{U}) \subset \mathcal{X} \xrightarrow{F} \mathcal{U} \times X_{b_0}$$

for a neighborhood $\mathcal{U}$ of b_0. Set $G = F^{-1}$. For any curve $\mu\colon [0,1] \to B$ such that $\mu(0) = b_0$, $\mu(1) = b_1$ we get a diffeomorphism:

$$f_\mu\colon X = X_{b_0} \to X_{b_1}.$$

This gives rise to isomorphisms

$$f_\mu^*\colon H^k(X_{b_1}, F) \to H^k(X_{b_0}, F), \tag{1.29}$$

where $F = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. In particular, since these isomorphisms depend only on the homotopy class of μ, we get a representation of $\pi_1(B, b_0)$ on $H^k(X_{b_0}, F)$ for $F = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. We will denote by $\mathbb{H}^k \to B$ the holomorphic vector bundle associated with this representation of $\pi_1(B, b_0)$. The fiber of $\mathbb{H}^k$ over $b \in B$ is isomorphic to $H^k(X_b, \mathbb{C})$ and the flat connection is known as the *Gauss-Manin connection.* Given $\alpha \in H^k(X_{b_0}, F)$, the section

$$\hat{\alpha}(t) \;:=\; G(t, \alpha)\;; \quad t \in \mathcal{U}$$

is a holomorphic, flat section over U.

The local system of flat sections agrees with is the k-th *direct image sheaf* $R^k\varphi_*F$. We recall that $R^k\varphi_*F$ is the sheaf associated with the presheaf that assigns to an open set U the cohomology $H^k(\varphi^{-1}(U), F)$. In our case, we may assume, without loss of generality, that the map

$$\mathrm{pr}_2 \circ F : \varphi^{-1}(U) \to X$$

deduced from (1.2) is a deformation retract. Hence, for U contractible,

$$H^k(\varphi^{-1}(U), F) \cong H^k(X, F).$$

2. Variations of Hodge Structure

2.1. **Geometric Variations of Hodge Structure.** We consider a family $\varphi\colon \mathcal{X} \to B$ and assume that $\mathcal{X} \subset \mathbb{P}^N$ so that each fiber X_t, $t \in B$, is now a smooth projective variety.† The Chern class of the hyperplane bundle restricted to $\mathcal{X}$ induces integral Kähler classes $\omega_t \in H^{1,1}(X_t) \cap H^2(X_t, \mathbb{Z})$ which fit together to define a section of the local system $R^2\varphi_*\mathbb{Z}$ over B.

On each fiber X_t we have a Hodge decomposition:

$$H^k(X_t, \mathbb{C}) \;=\; \bigoplus_{p+q=k} H^{p,q}(X_t),$$

where $H^{p,q}(X_t)$ is the space of de Rham cohomology classes and

$$H^{p,q}(X_t) \;\cong\; H^{p,q}_{\bar\partial}(X_t) \;\cong\; H^q(X_t, \Omega^p_{X_t}),$$

where the last term is the sheaf cohomology of X_t with values on the sheaf of holomorphic p-forms $\Omega^p_{X_t}$.

Theorem 2.1. *The Hodge numbers $h^{p,q}(X_t) = \dim_{\mathbb{C}} H^{p,q}(X_t)$ are constant.*

Proof. Recall that $H^{p,q}_{\bar\partial}(X_t) \cong \mathcal{H}^{p,q}(X_t)$, the $\bar\partial$-harmonic forms of bidegree (p,q). The Laplacian $\Delta^{X_t}_{\bar\partial}$ varies smoothly with the parameter t and consequently the dimension of its kernel is upper semicontinuous on t. This follows from the ellipticity of the Laplacian [47, Theorem 4.13]. Hence,

$$\dim_{\mathbb{C}} \mathcal{H}^{p,q}(X_t) \le \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X_{t_0}), \tag{2.1}$$

for t in a neighborhood of t_0. But, on the other hand,

$$\sum_{p+q=k} \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X_t) = b^k(X_t) = b^k(X_{t_0}) = \sum_{p+q=k} \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X_{t_0})$$

since X_t is diffeomorphic to X_{t_0}. Hence, $\dim_{\mathbb{C}} \mathcal{H}^{p,q}(X_t)$ must be constant. □

Recall from [7, Definition A.6] that the Hodge decomposition on $H^k(X_t, \mathbb{C})$ may be described by the filtration

$$F^p(X_t) \;:=\; \bigoplus_{a \ge p} H^{a,k-a}(X_t), \tag{2.2}$$

which satisfies the condition $H^k(X_t, \mathbb{C}) = F^p(X_t) \oplus \overline{F^{k-p+1}(X_t)}$. We set $f^p = \sum_{a \ge p} h^{a,k-a}$. Assume now that B is contractible and that $\mathcal{X}$ is C^∞-trivial over B. Then we have diffeomorphisms $g_t \colon X = X_{t_0} \to X_t$ which induce isomorphisms

$$g_t^* \colon H^k(X_t, \mathbb{C}) \to H^k(X, \mathbb{C}).$$

This allows us to define a map‡

$$\mathcal{P}^p \colon B \to G(f^p, H^k(X, \mathbb{C}))\;; \quad \mathcal{P}^p(t) = g_t^*(F^p(X_t)). \tag{2.3}$$

A Theorem of Kodaira (cf. [45, Proposition 9.22] implies that, since the dimension is constant, the spaces of harmonic forms $\mathcal{H}^{p,q}(X_t)$ vary smoothly with t. Hence the map $\mathcal{P}^p$ is smooth. In fact we have:

†Much of what follows holds with weaker assumptions. Indeed, it is enough to assume that a fiber X_t is Kähler to deduce that it will be Kähler for parameters close to t. We will not deal with this more general situation here and refer to [45, §9.3] for details.

‡Voisin [45] refers to the map $\mathcal{P}^p$ as the *period* map. To avoid confusion we will reserve this name for the map that assings to $t \in B$ the flag of subspaces $g_t^*(F^p(X_t))$.

Theorem 2.2. *The map $\mathcal{P}^p$ is holomorphic.*

Proof. In order to prove Theorem 2.2 we need to understand the differential of $\mathcal{P}^p$. For simplicity we will assume that $B = \Delta = \{z \in \mathbb{C} : |z| < 1\}$ though the results apply with minimal changes in the general case. Suppose then that $\varphi\colon \mathcal{X} \to \Delta$ is an analytic family, $X = X_0 = \varphi^{-1}(0)$, and we have a trivialization

$$F\colon \mathcal{X} \to \Delta \times X.$$

Set $G = F^{-1}$ and by $g_t\colon X \to X_t = \varphi^{-1}(t)$ the restriction $G|_{\{t\}\times X}$.

Then

$$\mathcal{P}^p(t) \;=\; F^p(t) \;:=\; g_t^*(F^p(X_t)) \subset H^k(X,\mathbb{C}),$$

and its differential at $t = 0$ is a linear map:

$$\mathcal{P}^p_{*,0}\colon T_0(\Delta) \to \operatorname{Hom}(F^p(0), H^k(X,\mathbb{C})/F^p(0)). \tag{2.4}$$

The assertion that $\mathcal{P}^p$ is holomorphic is equivalent to the statement that

$$\mathcal{P}^p_{*,0}\left(\frac{\partial}{\partial \bar{t}}\right) \;=\; 0. \tag{2.5}$$

We now describe the map (2.4) explicitly. Let $\alpha \in F^p(0)$. Since the subbundle $\mathbb{F}^p \subset \mathbb{H}^k$ is C^∞, we can construct a smooth section σ of $\mathbb{F}^p$ over $U \subset \Delta$, $0 \in U$, so that $\sigma(0) = \alpha$. Note that for $t \in U$,

$$\sigma(t) \in F^p(X_t) \subset H^k(X_t,\mathbb{C}),$$

and, consequently, we may view $g_t^*(\sigma(t))$ as a curve in $H^k(X,\mathbb{C})$ such such that $g_t^*(\sigma(t)) \in F^p(t)$. Then,

$$\left(\mathcal{P}^p_{*,0}\left(\frac{\partial}{\partial \bar{t}}\right)\right)(\alpha) \;=\; \left[\frac{\partial g_t^*(\sigma(t))}{\partial \bar{t}}\right] \mod F^p(0), \tag{2.6}$$

where $\partial/\partial\bar{t}$ acts on the coefficients of the forms $g_t^*(\sigma(t))$. Alternatively, we may regard this action as the pull-back of the covariant derivative

$$\nabla_{\partial/\partial\bar{t}}(\sigma). \tag{2.7}$$

We can realize the cohomology classes $\sigma(t)$ as the restriction of a global form

$$\Theta \in \bigoplus_{a\geq p} \mathcal{A}^{a,k-a}(\mathcal{X}) \tag{2.8}$$

such that $d(\Theta|_{X_t}) = 0$ and $\sigma(t) = [\Theta|_{X_t}] \in H^k(X_t,\mathbb{C})$ [45, Proposition 9.2.2]. We can now write the form $G^*(\Theta) \in \mathcal{A}^k(\Delta \times X,\mathbb{C})$ as:

$$G^*(\Theta) = \psi + dt \wedge \phi,$$

where neither ψ nor ϕ, involve dt, $d\bar{t}$; i.e. they are smooth forms on X whose coefficients vary smoothly with t. Note, in particular, that $\psi_{\{t\}\times X} = g_t^*(\Theta|_{X_t})$ is a closed form. We then have:

$$dG^*(\Theta) = dt \wedge \frac{\partial\psi}{\partial t} + d\bar{t} \wedge \frac{\partial\psi}{\partial\bar{t}} + dt \wedge d\phi. \tag{2.9}$$

Hence,

$$\frac{\partial g_t^*(\sigma(t))}{\partial\bar{t}} = \iota_{\partial/\partial\bar{t}}\left(dG^*(\Theta)\right). \tag{2.10}$$

Now, in view of (2.8), we have that at $t = 0$, ψ restricts to a closed form whose cohomology class lies in $F^p(X)$ and, therefore, (2.10) vanishes modulo $F^p(X)$. □

The interpretation of the differential of $\mathcal{P}^p$ in terms of the Gauss-Manin connection as in (2.7) and the expression (2.9) allow us to obtain a deeper statement:

Theorem 2.3 (Griffiths' Horizontality). *Let $\varphi\colon \mathcal{X} \to B$ be an analytic family and let $(\mathbb{H}^k, \nabla)$ denote the holomorphic vector bundle with the (flat) Gauss-Manin connection. Let $\sigma \in \Gamma(B, \mathbb{F}^p)$ be a smooth section of the holomorphic subbundle $\mathbb{F}^p \subset \mathbb{H}^k$. Then, for any $(1,0)$ vector field V on B,*

$$\nabla_V(\sigma) \in \Gamma(B, \mathbb{F}^{p-1}) \tag{2.11}$$

Proof. Again, for simplicity, we consider the case $B = \Delta$. Then, arguing as in the proof of Theorem 2.2, we have

$$\nabla_{\frac{\partial}{\partial t}|_{t=0}}(\sigma) = \frac{\partial}{\partial t}|_{t=0}\left(g_t^*(\sigma(t))\right).$$

But, (2.9) implies that:

$$\frac{\partial}{\partial t}|_{t=0}\left(g_t^*(\sigma(t))\right) = \iota_{\partial/\partial t}(dG^*(\Theta))|_{t=0} - d\phi|_{t=0}.$$

Since, clearly, the right-hand side lies in $F^{p-1}(0)$, the result follows. □

Remark 2. Given the Dolbeault isomorphism $H^{p,q}_{\bar\partial}(X) \cong H^q(X, \Omega^p)$, we can represent the differential of $\mathcal{P}^p$:

$$\mathcal{P}^p_{*,0}\colon T^h_0(B) \to \mathrm{Hom}(H^q(X_0, \Omega^p), H^{q+1}(X_0, \Omega^{p-1})$$

as the composition of the Kodaira-Spencer map:

$$\rho\colon T^h_0(B) \to H^1(X, T^h(X_0))$$

with the map

$$H^1(X, T^h(X_0)) \to \mathrm{Hom}(H^q(X_0, \Omega^p), H^{q+1}(X_0, \Omega^{p-1})$$

given by interior product and the product in Cech-cohomology. We refer to [45, Theorem 10.4] for details.

Given a a family $\varphi\colon \mathcal{X} \to B$ with $\mathcal{X} \subset \mathbb{P}^N$, the Chern class of the hyperplane bundle restricted to $\mathcal{X}$ induces integral Kähler classes $\omega_t \in H^{1,1}(X_t) \cap H^2(X_t, \mathbb{Z})$ which fit together to define a section of the local system $R^2\varphi_*\mathbb{Z}$ over B. This means that cup product by powers of the Kähler classes is a flat morphism and, consequently, the restriction of the Gauss-Manin connection to the primitive cohomology remains flat. Similarly, the polarization forms are flat and they polarize the Hodge decompositions on each fiber $H^k_0(X_t, \mathbb{C})$.

2.2. **Abstract Variations of Hodge Structure.**

The geometric situation described in 2.1 may be abstracted in the following definition:

Definition 2.4. Let B be a connected complex manifold, a *variation of Hodge structure* of weigth k (VHS) over B consists of a local system $\mathcal{V}_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules[†]

[†]In other contexts one only assumes the existence of a local system $\mathcal{V}_{\mathbb{Q}}$ (resp. $\mathcal{V}_{\mathbb{R}}$) of vector spaces over $\mathbb{Q}$ (resp. over $\mathbb{R}$) and refers to the resulting structure as a *rational* (resp. *real*) variation of Hodge structure.

and a filtration of the associated holomorphic vector bundle $\mathbb{V}$:

$$\cdots \subset \mathbb{F}^p \subset \mathbb{F}^{p-1} \subset \cdots \tag{2.12}$$

by holomorphic subbundles $\mathbb{F}^p$ satisfying:

i) $\mathbb{V} \;=\; \mathbb{F}^p \oplus \overline{\mathbb{F}^{k-p+1}}$ as C^∞ bundles, where the conjugation is taking relative to the local system of real vectorspaces $\mathcal{V}_{\mathbb{R}} :=\; \mathcal{V}_{\mathbb{Z}} \otimes \mathbb{R}$.
ii) $\nabla(\mathcal{F}^p) \subset \Omega^1_B \otimes \mathcal{F}^{p-1}$, where ∇ denotes the flat connection on $\mathbb{V}$, and $\mathcal{F}^p$ denotes the sheaf of holomorphic sections of $\mathbb{F}^p$.

We will refer to the holomorphic subbundles $\mathbb{F}^p$ as the *Hodge bundles* of the variation. It follows from [7, §A.2] that for each $t \in B$, we have a Hodge decomposition:

$$\mathbb{V}_t \;=\; \bigoplus_{p+q=k} \mathbb{V}_t^{p,q}\;; \quad \mathbb{V}_t^{q,p} = \overline{\mathbb{V}_t^{p,q}}, \tag{2.13}$$

where $\mathbb{V}^{p,q}$ is the C^∞ subbundle of $\mathbb{V}$ defined by:

$$\mathbb{V}^{p,q} = \mathbb{F}^p \cap \overline{\mathbb{F}^q}.$$

We will say that a VHS $(\mathbb{V}, \nabla, \{\mathbb{F}^p\})$ is *polarized* if there exists a flat non-degenerate bilinear form $\mathcal{Q}$ of parity $(-1)^k$ on $\mathbb{V}$, defined over $\mathbb{Z}$, such that for each $t \in B$ the Hodge structure on $\mathbb{V}_t$ is polarized, in the sense of [7, Definition A.9], by $\mathcal{Q}_t$.

We note that we can define a flat Hermitian form on $\mathbb{F}$ by $\mathcal{Q}^h(\cdot,\cdot) := i^{-k}\,\mathcal{Q}(\cdot,\bar{\cdot})$ making the decomposition (2.13) orthogonal and such that $(-1)^p\,\mathcal{Q}^h$ is positive definite on $\mathbb{V}^{p,k-p}$. The (generally not flat) positive definite Hermitian form on $\mathbb{V}$:

$$\mathcal{H} \;:=\; \sum_{p+q=k} (-1)^p \mathcal{Q}^h|_{\mathbb{V}^{p,q}}$$

is usually called the *Hodge metric* on $\mathbb{V}$.

We may then restate Theorems 2.2 and 2.3 together with the Hodge-Riemann bilinear relations as asserting that given a family $\varphi : \mathcal{X} \to B$ of smooth projective varieties, the holomorphic bundle whose fibers are the primitive cohomology $H^k_0(X_t, \mathbb{C})$, $t \in B$, endowed with the flat Gauss-Manin connection, carry a polarized Hodge structure of weight k.

3. Classifying Spaces

In analogy with the case of a family of projective varieties, we may regard a variation of Hodge structure as a family of Hodge structures on a fixed vector space $\mathbb{V}_{t_0}$. This is done via parallel translation relative to the flat connection ∇ and the result is well defined modulo the homotopy group $\pi_1(B, t_0)$.

In what follows we will fix the following data:

i) A lattice $V_{\mathbb{Z}}$. We will denote by $V_F = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} F$, for $F = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.
ii) An integer k.
iii) A collection of *Hodge numbers* $h^{p,q}$, $p + q = k$, such that $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = \dim_{\mathbb{C}} V_{\mathbb{C}}$. We set

$$f^p = \sum_{p \geq a} h^{a,k-a}.$$

iv) An integral, non degenerate bilinear form Q, of parity $(-1)^k$.

Definition 3.1. The space $D = D(V_{\mathbb{Z}}, Q, k, \{h^{p,q}\})$ consisting of all Hodge structures of weight k and Hodge numbers $h^{p,q}$, polarized by Q is called the *classifying space of Hodge structures weight* k and Hodge numbers $\{h^{p,q}\}$.

We will also be interested in considering the space $\check{D}$ consisting of all filtrations of $V_{\mathbb{C}}$:

$$\cdots \subset F^p \subset F^{p-1} \subset \cdots \tag{3.1}$$

such that $\dim_{\mathbb{C}} F^p = f^p$ and

$$Q(F^p, F^{k-p+1}) \;=\; 0. \tag{3.2}$$

We will refer to $\check{D}$ as the *dual* of D.

Example 3.2. A Hodge structure of weight 1 is a complex structure on $\mathbb{V}_{\mathbb{R}}$; that is, a decompostion $V_{\mathbb{C}} = \Omega \oplus \bar{\Omega}$. The polarization form Q is a non-degenerate alternating form and the polarization conditions reduce to:

$$Q(\Omega, \Omega) = 0\ ; \quad iQ(u, \bar{u}) > 0 \text{ if } 0 \neq u \in \Omega.$$

Hence, the classifying space for Hodge structures of weight one is the Siegel upper-half space defined in [7, Example 1.16]. The dual $\check{D}$ agrees with the space M defined in that same example. Geometrically, the weight-one case correspond to the study of the Hodge structure in the cohomology $H^1(X, \mathbb{C})$ for a smooth algebraic curve X. This example will be discussed from that point of view in Carlson's course.

The space $\check{D}$ may be regarded as the set of points in the product of Grassmannians

$$\prod_{p=1}^{k} G(f^p, V_{\mathbb{C}})$$

satisfying the flag compatibility conditions and the polynomial condition (3.2). Hence, $\check{D}$ is a projective variety. In fact we have:

Theorem 3.3. *Both D and $\check{D}$ are smooth complex manifolds. Indeed, $\check{D}$ is a homogeneous space $\check{D} \cong G_{\mathbb{C}}/B$, where*

$$G_{\mathbb{C}} \;:=\; \mathrm{Aut}(V_{\mathbb{C}}, Q); \tag{3.3}$$

that is, the group of of elements in $\mathrm{GL}(V, \mathbb{C})$ that preserve the non-degenerate bilinear form Q, and $B \subset G$ is the subgroup preserving a given flag $F_0 := \{F_0^p\}$. The open subset D of $\check{D}$ is the orbit of the real group $G = \mathrm{Aut}(V_{\mathbb{R}}, Q)$ and $D \cong G/V$, where

$$V = G \cap B$$

is a compact subgroup.

Proof. The fact that $G_{\mathbb{C}}$ acts transitively on $\check{D}$ is a linear algebra statement. We refer to [23, Theorem 4.3] for a proof. Being homogeneous, $\check{D}$ is smooth and since D is open in $\check{D}$, it is smooth as well. We illustrate these results in the cases of weight one and two.

For $k = 1$, $\dim_{\mathbb{C}} V_{\mathbb{C}} = 2n$, and $G_{\mathbb{C}} \cong \mathrm{Sp}(n, \mathbb{C})$. It follows from the non-degeneracy of Q that given any n-dimensional subspace $\Omega \in V_{\mathbb{C}}$ such that $Q(\Omega, \Omega) = 0$; i.e. a maximal isotropic subspace of $V_{\mathbb{C}}$, that there exists a basis $\{w_1, \ldots, w_{2n}\}$ of $V_{\mathbb{C}}$ such that $\{w_1, \ldots, w_n\}$ is a basis of Ω and, in this basis, the form Q is:

$$Q \;=\; \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}. \tag{3.4}$$

This shows that $G_{\mathbb{C}}$ acts transitively on $\check{D}$. On the other hand, if $\Omega_0 \in D$, then we can choose our basis so that $w_{n+i} = \bar{w}_i$ and, consequently, the group of real transformations $G \cong \mathrm{Sp}(n, \mathbb{R})$ acts transitively on D. The isotropy subgroup at some point $\Omega_0 \in D$ consists of real transformations in $\mathrm{GL}(V_{\mathbb{R}}) \cong \mathrm{GL}(2n, \mathbb{R})$ which preserve a complex structure and a Hermitian form in the resulting n-dimensional complex vector space. Hence $V \cong \mathrm{U}(n)$ and

$$D \;\cong \mathrm{Sp}(n, \mathbb{R})/\mathrm{U}(n).$$

In the weight-two case, $\dim V = 2h^{2,0} + h^{1,1}$ and Q is a non-degenerate symmetric form defined over $\mathbb{R}$ (in fact, over $\mathbb{Z}$). The complex Lie group $G \cong \mathrm{O}(2h^{2,0} + h^{1,1}, \mathbb{C})$. Given a reference polarized Hodge structure

$$V_{\mathbb{C}} = V_0^{2,0} \oplus V_0^{1,1} \oplus V_0^{0,2}; \quad V_0^{0,2} = \overline{V_0^{2,0}}$$

the real vector space $V_{\mathbb{R}}$ decomposes as:

$$V_{\mathbb{R}} = \Big((V_0^{2,0} \oplus V_0^{0,2}) \cap V_{\mathbb{R}} \Big) \oplus \Big(V_0^{1,1} \cap V_{\mathbb{R}} \Big) \tag{3.5}$$

and the form Q is negative definite on the first summand and positive definite on the second. Hence $G \cong \mathrm{O}(2h^{2,0}, h^{1,1})$. On the other hand, the elements in G that fix the reference Hodge structure must preserve each summand of (3.5). In the first summand, they must, in addition, preserve the complex structure $V_0^{2,0} \oplus V_0^{0,2}$ and a (negative) definite Hermitian form while, on the second summand, they must preserve a positive definite real symmetric form. Hence:

$$V \cong \mathrm{U}(h^{2,0}) \times \mathrm{O}(h^{1,1}).$$

Clearly the connected component of G acts transitively as well. These arguments generalize to arbitrary weight. □

Exercise 5. Describe the groups G and V for arbitrary even and odd weights.

The tangent bundles of the homogeneous spaces D and $\check{D}$ may be described in terms of Lie algebras. Let $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) denote the Lie algebra of $G_{\mathbb{C}}$ (resp G). Then

$$\mathfrak{g} = \{X \in \mathfrak{gl}(V_{\mathbb{C}}) : Q(Xu, v) + Q(u, Xv) = 0 \text{ ,for all } u, v \in \mathbb{V}_C\}, \tag{3.6}$$

and $\mathfrak{g}_0 = \mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{R}})$. The choice of a reference Hodge filtration $F_0 := \{F_0^p\}$ defines a filtration on $\mathfrak{g}$:

$$F^a\mathfrak{g} \;:=\; \{X \in \mathfrak{g} : X(F_0^p) \subset F_0^{p+a}\}.$$

We may, and will, assume that $F_0 \in D$; in particular, the filtration $F^a\mathfrak{g}$ defines a Hodge structure of weight 0 on $\mathfrak{g}$:

$$\mathfrak{g}^{a,-a} \;:=\; \{X \in \mathfrak{g} : X(V_0^{p,q}) \subset V_0^{p+a,q-a}\} \;=\; F^a\mathfrak{g} \cap F^{-a}\mathfrak{g}. \tag{3.7}$$

Note that $[F^p\mathfrak{g}, F^q\mathfrak{g}] \subset F^{p+q}\mathfrak{g}$ and $[\mathfrak{g}^{a,-a}, \mathfrak{g}^{b,-b}] \subset \mathfrak{g}^{a+b,-a-b}$. The Lie algebra $\mathfrak{b}$ of B is the subalgebra $F^0\mathfrak{g}$ and the Lie algebra of V is given by:

$$\mathfrak{v} \;=\; \mathfrak{g}_0 \cap \mathfrak{b} \;=\; \mathfrak{g}^{0,-0} \cap \mathfrak{g}_0.$$

Since $\check{D} = G_{\mathbb{C}}/B$ and B is the stabilizer of F_0, the holomorphic tangent space of $\check{D}$ at F_0 is $\mathfrak{g}/\mathfrak{b}$, while the tangent space at any other point is obtained via the action of G. More precisely, the holomorphic tangent bundle of $\check{D}$ is a homogeneous vector bundle associated to the principal bundle

$$B \to G_{\mathbb{C}} \to \check{D}$$

by the adjoint action of B on $\mathfrak{g}/\mathfrak{b}$. In other words,

$$T^h(\check{D}) \;\cong\; \check{D} \times_B \mathfrak{g}/\mathfrak{b}.$$

Since $[F^0\mathfrak{g}, F^p\mathfrak{g}] \subset F^p\mathfrak{g}$, it follows that the adjoint action of B leaves invariant the subspaces $F^p\mathfrak{g}$. In particular, we can consider the homogeneous subbundle $T^{-1,1}(\check{D})$ of $T^h(\check{D})$ associated with the subspace

$$F^{-1}\mathfrak{g} \;=\; \mathfrak{b} \oplus \mathfrak{g}^{-1,1}.$$

We will refer to $T^{-1,1}(\check{D})$ as the *horizontal* subbundle. Since $D \subset \check{D}$ is open, these bundles restrict to holomorphic bundles over D.

It will be useful to unravel the definition of the horizontal bundle. We may view an element of the fiber of $T^{-1,1}(\check{D})$ as the equivalence class of a pair $(F, [X])$, where $[X] \in \mathfrak{g}/\mathfrak{b}$. Then, if $F = g \cdot F_0$, we have that $Ad(g^{-1})(X) \in F^{-1}\mathfrak{g}$, and, if we regard $\mathfrak{g}$ as a Lie algebra of endomorphisms of $V_{\mathbb{C}}$ this implies

$$(g^{-1} \cdot X \cdot g)F_0^p \subset F_0^{p-1}$$

or, equivalently

$$X(F^p) \subset F^{p-1}. \tag{3.8}$$

We may now define the period map of an abstract variation of Hodge structure. Let $(\mathbb{V}, \nabla, \mathcal{Q}, \{\mathbb{F}^p\})$ be a polarized variation of Hodge structure over a connected, complex manifold B and let $b_0 \in B$. Given a curve $\mu\colon [0,1] \to B$ with $\mu(0) = b_0$ and $\mu(1) = b_1$ we may define a $\mathbb{C}$-linear isomorphism

$$\mu^*\colon \mathbb{V}_{b_1} \to \mathbb{V}_{b_0}$$

by parallel translation relative to the flat connection ∇. These isomorphisms depend on the homotopy class of μ and, as before, we denote by

$$\rho\colon \pi_1(B, b_0) \to \mathrm{GL}(\mathbb{V}_{b_0})$$

the resulting representation. We call ρ the *monodromy representation* and the image

$$\Gamma \;:=\; \rho(\pi_1(B, b_0)) \subset \mathrm{GL}(\mathbb{V}_{b_0}, \mathbb{Z}) \tag{3.9}$$

the *monodromy subgroup*. We note that since $\mathcal{V}_{\mathbb{Z}}$ and $\mathcal{Q}$ are flat, the monodromy representation is defined over $\mathbb{Z}$ and preserves the bilinear form $\mathcal{Q}_{b_0}$. In particular, since V is compact, the action of Γ on D is properly discontinuous and the quotient D/Γ is an analytic variety.

Hence, we may view the polarized Hodge structures on the fibers of $\mathbb{V}$ as a family of polarized Hodge structures on $\mathbb{V}_{b_0}$ well-defined up to the action of the monodromy subgroup. That is, we obtain a map

$$\Phi\colon B \to D/\Gamma, \tag{3.10}$$

where D is the appropriate classifying space for polarized Hodge structures. We call Φ the *period map* of the polarized VHS.

Theorem 3.4. *The period map has local liftings to D which are holomorphic. Moreover, the differential takes values on the horizontal subbundle $T^{-1,1}(D)$.*

Proof. This is just the statement that the subbundles $\mathbb{F}^p$ are holomorphic together with condition ii) in Definition 2.4. □

We will refer to any locally liftable map $\Phi\colon B \to D/\Gamma$ with holomorphic and horizontal local liftings as a period map.

Example 3.5. In the weight-one case, $T^{-1,1}(D) = T^h(D)$. Hence, a period map is simply a locally liftable, holomorphic map $\Phi\colon B \to D/\Gamma$, where D is the Siegel upper half space and $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$ is a discrete subgroup. If $\tilde{B} \to B$ is the universal covering of B, we get a global lifting

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{\Phi}} & D \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\Phi} & D/\Gamma \end{array}$$

and the map $\tilde{\Phi}$ is a holomorphic map with values in Siegel's upper half-space.

Exercise 6. Describe period maps in the weight-two case.

4. Mixed Hodge Structures and the Orbit Theorems

In the remainder of these notes we will be interested in studying the asymptotic behavior of a variation of Hodge structure of weight k. Geometrically, this situation arises, for example, when we have a family of smooth projective varieties $\mathcal{X} \to B$, where B is a quasi-projective variety defined as the complement of a divisor with normal crossings Y in a smooth projective variety $\bar{B}$. Then, locally on the divisor Y, we may consider the polarized variation of Hodge structure defined by the primitive cohomology over an open set $U \subset \bar{B}$ such that $U = \Delta^n$ and

$$U \cap Y = \{z \in \Delta^n : z_1 \cdots z_r = 0\}.$$

This means that

$$U \cap B = (\Delta^*)^r \times \Delta^{n-r}.$$

Thus, we will consider period maps

$$\Phi\colon (\Delta^*)^r \times \Delta^{n-r} \to D/\Gamma \tag{4.1}$$

and their liftings to the universal cover:

$$\tilde{\Phi}\colon H^r \times \Delta^{n-r} \to D, \tag{4.2}$$

where $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the universal covering space of Δ^* as in Example 1.5. The map $\tilde{\Phi}$ is then holomorphic and horizontal.

We will denote by $c_1, \ldots, c_r$ the generators of $\pi_1((\Delta^*)^r)$; i.e., c_j is a clockwise loop around the origin in the j-th factor Δ^*. Let $\gamma_j = \rho(c_j)$. Clearly the monodromy transformations γ_j, $j = 1, \ldots, r$, commute. We have:

Theorem 4.1 (Monodromy Theorem). *The monodromy transformations γ_j, $j = 1, \ldots, r$, are quasi unipotent; that is, there exist integers ν_j such that $(\gamma_j^{\nu_j} - \mathrm{id})$ is nilpotent. Moreover, the index of nilpotency of $(\gamma_j^{\nu_j} - \mathrm{id})$ is at most $k+1$.*

Proof. In the geometric case this result is due to Landman[36]. The proof for (integral) variations of Hodge structure is due to Borel (cf. [41, (4.5)]). The statement on the index of nilpotency is proved in [41, (6.1)] □

4.1. **Nilpotent Orbits.** For simplicity of notation, we will often assume that $r = n$. This will, generally, entail no loss of generality as our statements will usually hold uniformly on compact subsets of Δ^{n-r} but this will be made precise when necessary. We will also assume that the monodromy transformations γ_j are actually *unipotent*, that is $\nu_j = 1$. This may be accomplished by lifting the period map to a finite covering of $(\Delta^*)^r$. We point out that most of the results that follow hold for real variations of Hodge structure, provided that we assume that the monodromy transformations are unipotent. In what follows we will write:

$$\gamma_j = e^{N_j} \ ; \quad j = 1, \ldots, r, \tag{4.3}$$

where N_j are nilpotent elements in $\mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$ such that $N^{k+1} = 0$. We then have

$$\tilde{\Phi}(z_1, \ldots, z_j + 1, \ldots, z_r) \ = \ \exp(N_j) \cdot \tilde{\Phi}(z_1, \ldots, z_j, \ldots, z_r), \tag{4.4}$$

and the map $\Psi \colon H^r \to \check{D}$ defined by

$$\Psi(z_1, \ldots, z_r) \ := \ \exp\left(- \sum_{j=1}^{r} z_j \, N_j \right) \cdot \tilde{\Phi}(z_1, \ldots, z_r) \tag{4.5}$$

is the lifting of a holomorphic map $\psi \colon (\Delta^*)^r \to D$ so that

$$\psi(t_1, \ldots, t_r) \ = \ \Psi\left(\frac{\log t_1}{2\pi i}, \ldots, \frac{\log t_r}{2\pi i} \right). \tag{4.6}$$

Example 4.2. Let $F_0 \in \check{D}$ and let $N_1, \ldots, N_r$ be commuting elements in $\mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$ such that

$$N_j(F_0^p) \subset F_0^{p-1}. \tag{4.7}$$

Then the map

$$\theta \colon H^r \to \check{D} \ ; \quad \theta(z_1, \ldots, z_r) = \exp\left(\sum_{j=1}^{r} z_j \, N_j \right) \cdot F_0 \tag{4.8}$$

is holomorphic and, because of (4.7) and (3.8), its differential takes values on the horizontal subbundle. Hence, if we assume that there exists $\alpha > 0$ such that:

$$\theta(z_1, \ldots, z_r) \in D \ ; \quad \text{for} \quad \mathrm{Im}(z_j) > \alpha, \tag{4.9}$$

the map θ is the lifting of a period map defined on a product of punctured disks Δ_ε^*. Such a map will be called a nilpotent orbit. Note that for a nilpotent orbit the map (4.5) is constant, equal to F_0.

Theorem 4.3 (Nilpotent Orbit Theorem). *Let $\Phi\colon (\Delta^*)^r \times \Delta^{n-r} \to D$ be a period map and let $N_1, \dots, N_r$ be the monodromy logarithms. Let*

$$\psi\colon (\Delta^*)^r \times \Delta^{n-r} \to \check{D}$$

be as in (4.6). *Then*

i) *The map ψ extends holomorphically to $\Delta^r \times \Delta^{n-r}$.*

ii) *For each $w \in \Delta^{n-r}$, the map $\theta\colon \mathbb{C}^r \times \Delta^{n-r} \to \check{D}$ given by*

$$\theta(z,w) = \exp\left(\sum z_j\, N_j\right).\psi(0,w)$$

is a nilpotent orbit. Moreover, if $C \subset \Delta^{n-r}$ is compact, there exists $\alpha > 0$ such that $\theta(z,w) \in D$ for $\mathrm{Im}(z_j) > \alpha$, $1 \le j \le n$, $w \in C$.

iii) *For any G-invariant distance d on D, there exist positive constants β, K, such that, for $\mathrm{Im}(z_j) > \alpha$,*

$$d(\Phi(z,w), \theta(z,w)) \le K \sum_j (\mathrm{Im}(z_j))^\beta\, e^{-2\pi \mathrm{Im}(z_j)}\ .$$

Moreover, the constants α, β, K depend only on the choice of d and the weight and Hodge numbers used to define D and may be chosen uniformly for w in a compact subset $C \subset \Delta^{n-r}$.

Proof. The proof of Theorem 4.3, which is due to Wilfried Schmid [41], hinges upon the existence of G-invariant Hermitian metrics on D, whose holomorphic sectional curvatures along horizontal directions are negative and bounded away from zero [29]. We refer the reader to [30] for an expository account and to [42] for an enlightening proof in the case when D is Hermitian symmetric[†]; the latter is also explicitely worked out in [6] for VHS of weight one. We should remark that the distance estimate in iii) is stronger than that in Schmid's original version [41, (4.12)] and is due to Deligne (cf. [12, (1.15)] for a proof). □

The Nilpotent Orbit Theorem has a very nice interpretation in the context of Deligne's *canonical extension* [16]. Let $\mathbb{V} \to (\Delta^*)^r \times \Delta^{n-r}$ be the flat bundle underlying a polarized VHS and pick a base point (t_0, w_0). Given $v \in V := \mathbb{V}_{(t_0,w_0)}$, let $v^\flat$ denote the multivalued flat section of $\mathbb{V}$ defined by v. Then

$$\tilde{v}(t,w) \ :=\ \exp\left(\sum_{j=1}^r \frac{\log t_j}{2\pi i} N_j\right) \cdot v^\flat(t,w) \tag{4.10}$$

is a global section of $\mathbb{V}$. The canonical extension $\overline{\mathbb{V}} \to \Delta^n$ is characterized by its being trivialized by sections of the form (4.10). The Nilpotent Orbit Theorem is then equivalent to the regularity of the Gauss-Manin connection and implies that the Hodge bundles $\mathbb{F}^p$ extend to holomorphic subbundles $\overline{\mathbb{F}^p} \subset \overline{\mathbb{V}}$. Writing the Hodge bundles interms of a basis of sections of the form (4.10) yields the holomorphic map Ψ. Its constant part —corresponding to the nilpotent orbit— defines a polarized VHS as well. The connection ∇ extends to a connection on Δ^n with logarithmic poles along the divisor $\{t_1 \cdots t_r = 0\}$ and nilpotent residues.

[†]In this case the Nilpotent Orbit Theorem follows from the classical Schwarz Lemma.

Given a period map $\Phi\colon (\Delta^*)^r \to D/\Gamma$, we will call the value

$$F_{\lim} := \psi(0) \in \check{D}$$

the *limiting Hodge filtration.* Note that $F_{\lim}$ depends on the choice of coordinates in $(\Delta^*)^r$. Indeed, a change of coordinates compatible with the divisor structure must be, after relabeling if necessary, of the form $(\hat{t}_1, \ldots, \hat{t}_r) = (t_1 f_1(t), \ldots, t_r f_r(t))$ where f_j are holomorphic around $0 \in \Delta^r$ and $f_j(0) \neq 0$. We then have from (4.6):

$$\begin{aligned} \hat{\psi}(\hat{t}\,) &= \exp(-\frac{1}{2\pi i}\sum_{j=1}^{r}\log(\hat{t}_j)N_j)\cdot\Phi(\hat{t}\,) \\ &= \exp(-\frac{1}{2\pi i}\sum_{j=1}^{r}\log(f_j)N_j)\exp(-\frac{1}{2\pi i}\sum_{j=1}^{r}\log(t_j)N_j)\cdot\Phi(t) \\ &= \exp(-\frac{1}{2\pi i}\sum_{j=1}^{r}\log(f_j)N_j)\cdot\Psi(t), \end{aligned} \tag{4.11}$$

and, letting $t \to 0$

$$\hat{F}_{\lim} = \exp\left(-\frac{1}{2\pi i}\sum_{j}\log(f_j(0))N_j\right)\cdot F_{\lim}. \tag{4.12}$$

4.2. **Mixed Hodge Structures.** We will review some basic notions about mixed Hodge structures following the notation of [12]. We refer to [22] and [40] for a full account.

Definition 4.4. Let $V_{\mathbb{Q}}$ be a vector space over $\mathbb{Q}$, $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes \mathbb{R}$, and $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes \mathbb{C}$. A *mixed Hodge Structure* (MHS) on $V_{\mathbb{C}}$ consists of a pair of filtrations of V, (W, F), where W is an increasing filtration defined over $\mathbb{Q}$ and F is decreasing, such that F induces a Hodge structure of weight k on $\mathrm{Gr}_k^W := W_k/W_{k-1}$ for each k.

The filtration W is called the *weight filtration*, while F is called the *Hodge filtration*. We point out that for many of the subsequent results, it is enough to assume that W is defined over $\mathbb{R}$. This notion is compatible with passage to the dual and with tensor products. In particular, given a MHS on $V_{\mathbb{C}}$ we may define a MHS on $\mathfrak{gl}(V_{\mathbb{C}})$ by:

$$W_a\mathfrak{gl} := \{X \in \mathfrak{gl}(V_{\mathbb{C}}) : X(W_\ell) \subset W_{\ell+a}\} \tag{4.13}$$

$$F^b\mathfrak{gl} := \{X \in \mathfrak{gl}(V_{\mathbb{C}}) : X(F^p) \subset F^{p+b}\} \tag{4.14}$$

An element $T \in W_{2a}\mathfrak{gl} \cap F^a\mathfrak{gl} \cap \mathfrak{gl}(V_{\mathbb{Q}})$ is called an *(a,a)-morphism* of (W, F).

Definition 4.5. A *splitting* of a MHS (W, F) is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{p,q} J^{p,q}$$

such that

$$W_\ell = \bigoplus_{p+q\leq\ell} J^{p,q}\ ; \quad F^p = \bigoplus_{a\geq p} J^{a,b} \tag{4.15}$$

An (a,a) morphism T of a MHS (W,F) is said to be *compatible* with the splitting $\{J^{p,q}\}$ if $T(J^{p,q}) \subset J^{p+a,q+a}$.

Every MHS admits splittings compatible with all its morphisms. In particular, we have the following result due to Deligne[20]:

Theorem 4.6. *Given a MHS (W,F) the subspaces:*

$$I^{p,q} \;:=\; F^p \cap W_{p+q} \cap \left(\overline{F^q} \cap W_{p+q} + \overline{U^{q-1}_{p+q-2}}\right), \text{ with} \tag{4.16}$$

$$U^a_b = \sum_{j\geq 0} F^{a-j} \cap W_{b-j},$$

define a splitting of (W,F) compatible with all morphisms. Moreover, $\{I^{p,q}\}$ is uniquely characterized by the property:

$$I^{p,q} \;\equiv \overline{I^{q,p}} \quad \left(\bmod \bigoplus_{a<p;b<q} I^{a,b}\right) \tag{4.17}$$

This correspondence establishes an equivalence of categories between MHS and bigradings $\{I^{p,q}\}$ satisfying (4.17).

Proof. We refer to [12, Theorem 2.13] for a proof. □

Definition 4.7. A mixed Hodge structure (W,F) is said to split over $\mathbb{R}$ if it admits a splitting $\{J^{p,q}\}$ such that

$$J^{q,p} = \overline{J^{p,q}}.$$

In this case,

$$V_{\mathbb{C}} \;=\; \bigoplus_k \left(\bigoplus_{p+q=k} J^{p,q}\right)$$

is a decomposition of $V_{\mathbb{C}}$ as a direct sum of Hodge structures.

Example 4.8. The paradigmatic example of a mixed Hodge structure split over $\mathbb{R}$ is the Hodge decomposition on the cohomology of a compact Kähler manifold X (cf. [7, §5]). Let

$$V_{\mathbb{Q}} = H^*(X,\mathbb{Q}) = \bigoplus_{k=0}^{2n} H^k(X,\mathbb{Q})$$

and set

$$J^{p,q} = H^{n-p,n-q}(X).$$

Thus,

$$W_\ell \;=\; \bigoplus_{d\geq 2n-\ell} H^d(X,\mathbb{C}), \quad F^p \;=\; \bigoplus_s \bigoplus_{r\leq n-p} H^{r,s}(X). \tag{4.18}$$

With this choice of indexing, the operators L_ω, where ω is a Kähler class, are $(-1,-1)$-morphisms of the MHS.

The situation described by Example 4.8 carries additional structure: the Lefschetz theorems and the Hodge-Riemann bilinear relations. We extend these ideas to the case of abstract mixed Hodge structures. Recall from [7, Proposition A.12] that given a nilpotent transformation $N \in \mathfrak{gl}(V_{\mathbb{Q}})$ there exists a unique increasing filtration defined over $\mathbb{Q}$, $W = W_\ell(N)$, such that:

i) $N(W_\ell) \subset W_{\ell-2}$,
ii) For $\ell \geq 0 : N^\ell \colon \mathrm{Gr}^W_\ell \to \mathrm{Gr}^W_{-\ell}$ is an isomorphism.

Definition 4.9. A *polarized MHS* (PMHS) [5, (2.4)] of weight $k \in \mathbb{Z}$ on $V_{\mathbb{C}}$ consists of a MHS (W, F) on V, a $(-1,-1)$ morphism $N \in \mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$, and a nondegenerate, rational bilinear form Q such that:

i) $N^{k+1} = 0$,
ii) $W = W(N)[-k]$, where $W[-k]_\ell := W_{\ell-k}$,
iii) $Q(F^a, F^{k-a+1}) = 0$ and,
iv) the Hodge structure of weight $k+l$ induced by F on $\ker(N^{l+1} : \mathrm{Gr}^W_{k+l} \to \mathrm{Gr}^W_{k-l-2})$ is polarized by $Q(\cdot, N^l \cdot)$.

Example 4.10. We continue with Example 4.8. We may restate the Hard Lefschetz Theorem [7, Corollary 5.11] and the Hodge-Riemann bilinear relations [7, Theorem 5.17] by saying that the mixed Hodge structure in the cohomology $H^*(X, \mathbb{C})$ of an n-dimensional compact Kähler manifold X is a MHS of weight n polarized by the rational bilinear form Q on $H^*(X, \mathbb{C})$ defined by:

$$Q([\alpha], [\beta]) \;=\; (-1)^{r(r+1)/2} \int_X \alpha \wedge \beta \;; \quad [\alpha] \in H^r(X, \mathbb{C}), [\beta] \in H^s(X, \mathbb{C})$$

and the nilpotent operator L_ω for any Kähler class ω. Note that ii) in [7, Theorem 5.17] is the assertion that $L_\omega \in \mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$.

There is a very close relationship between polarized mixed Hodge structures and nilpotent orbits as indicated by the following:

Theorem 4.11. *Let $\theta(z) = \exp(\sum_{j=1}^r z_j N_j) \cdot F$ be a nilpotent orbit in the sense of Example 4.2, then:*

i) *Every element in the cone*

$$\mathcal{C} \;:=\; \{N = \sum_{j=1}^r \lambda_j N_j \;; \lambda_j \in \mathbb{R}_{>0}\} \subset \mathfrak{g} \tag{4.19}$$

defines the same weight filtration $W(\mathcal{C})$.

ii) *The pair $(W(\mathcal{C})[-k], F)$ defines a MHS polarized by every $N \in \mathcal{C}$.*
iii) *Conversely, suppose $\{N_1, \ldots, N_r\} \in \mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$ are commuting nilpotent elements with the property that the weight filtration $W(\sum \lambda_j N_j)$, is independent of the choice of $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{>0}$. Then, if $F \in \check{D}$ is such that $(W(\mathcal{C})[-k], F)$ is polarized by every[†] element $N \in \mathcal{C}$, the map $\theta(z) = \exp(\sum_{j=1}^r z_j N_j) \cdot F$ is a nilpotent orbit.*

[†]In fact, it suffices to assume that this holds for *some* $N \in \mathcal{C}$.

Proof. Part i) is proved in [5, (3.3)], while ii) was proved by Schmid [41, Theorem 6.16] as a consequence of his SL_2-orbit theorem to be discussed below. In the case of geometric variations it was also shown by Steenbrink [43] and Clemens and Schmid [14]. The converse is Proposition 4.66 in [12]. □

Remark 3. If $\{N_1, \ldots, N_r\}$ and F satisfy the conditions in iii) of Theorem 4.11 we will simply say that $(N_1, \ldots, N_r; F)$ is a nilpotent orbit. This notation emphasizes the fact that the notion of a nilpotent orbit is a (polarized) linear algebra notion.

Example 4.12. We continue with the situation discussed in Examples 4.8 and 4.10. Let $\omega_1, \ldots, \omega_r \in \mathcal{K}$ be Kähler classes in the compact Kähler manifold X. Then, clearly, the nilpotent transformations $L_{\omega_1}, \cdots, L_{\omega_r}$ commute and since every positive linear combination is also a Kähler class, it follows from the Hard Lefschetz Theorem that the weight filtration is independent of the coefficients. Moreover, the assumptions of iii) in Theorem 4.11 hold and therefore the map

$$\theta(z_1, \ldots, z_r) \ := \ \exp(z_1 L_{\omega_1} + \cdots + z_r L_{\omega_r}) \cdot F,$$

where F is as in (4.18) is a nilpotent orbit and hence defines a variation of Hodge structure on $H^*(X, \mathbb{C})$. Note that these Hodge structures are defined in the *total* cohomology of X. The relationship between this VHS and mirror symmetry is discused in [9, 10]. This PVHS on $H^*(X, \mathbb{C})$ plays a central role in the mixed Lefschetz and Hodge-Riemann bilinear relations discussed in [7, §5].

4.3. SL$_2$**-orbits.** Theorem 4.11 establishes a relationhip between polarized mixed Hodge structures and nilpotent orbits. In the case of PMHS split over $\mathbb{R}$ this correspondence yields an equivalence with a particular class of nilpotent orbits equivariant under a natural action of $SL(2, \mathbb{R})$. For simplicity, we will restrict ourselves to the one-variable case and refer the reader to [12, 11] for the general case.

Let (W, F_0) be a MHS on $V_{\mathbb{C}}$, split over $\mathbb{R}$ and polarized by $N \in F_0^{-1}\mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$. Since $W = W(N)[-k]$, the subspaces

$$V_\ell = \bigoplus_{p+q=k+\ell} I^{p,q}(W, F_0)\,, \qquad -k \le \ell \le k$$

constitute a grading of $W(N)$ defined over $\mathbb{R}$. Let $Y = Y(W, F_0)$ denote the real semisimple endomorphism of $V_{\mathbb{C}}$ which acts on V_ℓ as multiplication by the integer ℓ. Since $NV_\ell \subset V_{\ell-2}$,

$$[Y, N] = -2N. \tag{4.20}$$

Because N polarizes the MHS (W, F_0) one also obtains (cf. [12, (2.7)]):

$$Y \in \mathfrak{g}_0, \text{ and}$$

$$\text{There exists } N^+ \in \mathfrak{g}_0 \text{ such that } [Y, N^+] = 2N^+, [N^+, N] = Y. \tag{4.21}$$

Therefore, there is a Lie algebra homomorphism $\rho\colon \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ defined over $\mathbb{R}$ such that, for the standard generators $\{\mathbf{y}, \mathbf{n}_+, \mathbf{n}_-\}$ defined in [7, (A.30)]:

$$\rho(\mathbf{y}) = Y\,, \quad \rho(\mathbf{n}_-) = N\,, \quad \rho(\mathbf{n}_+) = N^+. \tag{4.22}$$

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ carries a Hodge structure of weight 0:

$$(\mathfrak{sl}(2, \mathbb{C}))^{-1,1} = \overline{(\mathfrak{sl}(2, \mathbb{C}))^{1,-1}} = \mathbb{C}(i\,\mathbf{y} + \mathbf{n}_- + \mathbf{n}_+)$$

$$(\mathfrak{sl}(2,\mathbb{C}))^{0,0} = \mathbb{C}(\mathbf{n}_+ - \mathbf{n}_-)\,.$$

A homomorphism $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ is said to be *Hodge* at $F \in D$, if it is a morphism of Hodge structures: that defined above on $\mathfrak{sl}(2,\mathbb{C})$ and the one determined by $F\mathfrak{g}$ in $\mathfrak{g}$. The lifting $\tilde\rho\colon \mathrm{SL}(2,\mathbb{C}) \to G_{\mathbb{C}}$ of such a morphism induces a horizontal, equivariant embedding $\hat\rho\colon \mathbb{P}^1 \longrightarrow \check D$ by $\hat\rho(g.i) = \rho(g).F$, $g \in \mathrm{SL}(2,\mathbb{C})$. Moreover

i) $\tilde\rho(SL(2,\mathbb{R})) \subset G_{\mathbb{R}}$ and, therefore $\hat\rho(H) \subset D$, where H is the upper-half plane.
ii) $\hat\rho(z) = (\exp z\rho(\mathbf{n}_-))(\exp(-i\rho(\mathbf{n}_-))).F$.
iii) $\hat\rho(z) = (\exp x\rho(\mathbf{n}_-))(\exp(-1/2)\log y\,\rho(\mathbf{y}))).F$. for $z = x+iy \in H$.

Theorem 4.13. *Let (W, F_0) be a MHS split over $\mathbb{R}$ and polarized by $N \in F_0^{-1}\mathfrak{g}$. Then*

i) *The filtration $F_{\sqrt{-1}} := \exp i\,N.F_0$ lies in D.*
ii) *The homomorphism $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ defined by (4.22) is Hodge at $F_{\sqrt{-1}}$.*

Conversely, if a homomorphism $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ is Hodge at $F \in D$, then

$$(W(\rho(\mathbf{n}_-))[-k], \exp(-i\rho(\mathbf{n}_-)).F)$$

is a MHS, split over $\mathbb{R}$ and polarized by $\rho(\mathbf{n}_-)$.

The following is a simplified version of Schmid's SL_2-orbit theorem. We refer to [41] for a proof.

Theorem 4.14 (SL_2**-Orbit Theorem).** *Let $z \mapsto \exp zN.F$ be a nilpotent orbit. There exists*

i) *A filtration $F_{\sqrt{-1}} \in D$;*
ii) *A homomorphism $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ Hodge at $F_{\sqrt{-1}}$;*
iii) *A real analytic, $G_{\mathbb{R}}$-valued function $g(y)$, defined for $y >> 0$,*

such that

i) *$N = \rho(\mathbf{n}_-)$;*
ii) *For $y >> 0$, $\exp(iyN).F = g(y)\,\exp(iyN).F_0$, where $F_0 = \exp(-iN).F_{\sqrt{-1}}$;*
iii) *Both $g(y)$ and $g(y)^{-1}$ have convergent power series expansions at $y = \infty$, of the form $1 + \sum_{n=1}^{\infty} A_n\,y^{-n}$, with*

$$A_n \in W_{n-1}\mathfrak{g} \cap \ker(\mathrm{ad}N)^{n+1}\,.$$

We may regard the SL_2-orbit theorem as associating to any given nilpotent orbit a distinguished nilpotent orbit, whose corresponding limiting mixed Hodge structure splits over $\mathbb{R}$, together with a very fine description of the relationship between the two orbits. In particular, it yields the fact that nilpotent orbits are equivalent to PMHS and, given this it may be interpreted as associating to any PMHS another one which splits over $\mathbb{R}$. One may reverse this process and take as a starting point the existence of the limiting MHS associated with a nilpotent orbit. It is then possible to define characterize functorially the PMHS corresponding to the SL_2-orbit. We refer to [12] for a full discussion. It is also possible to define other functorial real splittings of a MHS. One such is due to Deligne [20] and is central to the several-variable arguments in [12].

5. Asymptotic Behavior of a Period Mapping

In this section we will study the asymptotic behavior of a period map. Much of this material is taken from [11, 9, 10]. Our setting is the same as in the previous section; i.e. we consider a period map

$$\Phi\colon (\Delta^*)^r \times \Delta^{n-r} \to D/\Gamma$$

and its lifting to the universal cover:

$$\tilde{\Phi}\colon H^r \times \Delta^{n-r} \to D,$$

The map $\tilde{\Phi}$ is, thus, holomorphic and horizontal. We assume that the monodromy transformations $\gamma_1, \ldots, \gamma_r$ are unipotent and let $N_1, \ldots, N_r \in \mathfrak{g} \cap \mathfrak{gl}(V_{\mathbb{Q}})$ denote the monodromy logarithms. Let $F_{\lim}(w)$, $w \in \Delta^{n-r}$ be the limiting Hodge filtration. Then, for each $w \in \Delta^{n-r}$ we have a nilpotent orbit $(N_1, \ldots, N_r; F_{\lim}(w))$. Moreover, the Nilpotent Orbit Theorem implies that we may write:

$$\tilde{\Phi}(z, w) \;=\; \exp\left(\sum_{j=1}^{r} z_j N_j\right) \cdot \psi(t, w), \tag{5.1}$$

where $t_j = \exp(2\pi i z_j)$, and $\psi(t, w)$ is a holomorphic map on Δ^n with values on $\check{D}$ and $\psi(0, w) = F_{\lim}(w)$.

Since $\check{D}$ is a homogeneous space of the Lie group $G_{\mathbb{C}}$, we can obtain holomorphic liftings of ψ to $G_{\mathbb{C}}$. We describe a lifting adapted to the limiting mixed Hodge structure. Let $W = W(\mathcal{C})[-k]$ denote the shifted weight filtration of any linear combination of $N_1, \ldots, N_r$ with positive real coefficients, and let $F_0 = F_{\lim}(0)$. We let $\{I^{p,q}\}$ denote the canonical bigrading of the mixed Hodge structure (W, F_0) (cf. Theorem 4.6). The subspaces

$$I^{a,b}\mathfrak{g} \;:=\; \{X \in \mathfrak{g} : X(I^{p,q}) \subset I^{p+a,q+b}\} \tag{5.2}$$

define the canonical bigrading of the mixed Hodge structure defined by $(W\mathfrak{g}, F_0\mathfrak{g})$ on $\mathfrak{g}$. We note that

$$\left[I^{a,b}\mathfrak{g}, I^{a',b'}\mathfrak{g}\right] \subset I^{a+a',b+b'}\mathfrak{g}.$$

Set

$$\mathfrak{p}_a \;:=\; \bigoplus_q I^{a,q}\mathfrak{g} \quad \text{and} \quad \mathfrak{g}_- \;:=\; \bigoplus_{a \leq -1} \mathfrak{p}_a. \tag{5.3}$$

Since, by (4.15),

$$F_0^0(\mathfrak{g}) = \bigoplus_{p \geq 0} I^{p,q}\mathfrak{g}\,,$$

it follows that $\mathfrak{g}_-$ is a nilpotent subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{b} = F_0^0(\mathfrak{g})$, the lie algebra of the isotropy subgroup B of $G_{\mathbb{C}}$ at F_0. Hence, in a neighborhood of the origin in Δ^n, we may write:

$$\psi(t, w) \;=\; \exp(\Gamma(t, w)) \cdot F_0,$$

where

$$\Gamma\colon U \subset \Delta^n \to \mathfrak{g}_- \tag{5.4}$$

is holomorphic in an open set U around the origin, and $\Gamma(0) = 0$. Consequently, we may rewrite (5.1) as:

$$\tilde{\Phi}(t,w) \;=\; \exp\left(\sum_{j=1}^{r} z_j\, N_j\right) \cdot \exp(\Gamma(t,w)) \cdot F_0 \tag{5.5}$$

Since $N_j \in I^{-1,-1}\mathfrak{g} \subset \mathfrak{g}_-$, the product

$$E(z,w) \;:=\; \exp\left(\sum_{j=1}^{r} z_j\, N_j\right) \cdot \exp(\Gamma(t,w))$$

lies in the nilpotent group $\exp(\mathfrak{g}_-)$ and, hence we may write

$$E(z,w) \;:=\; \exp(X(z,w)) \;; \quad X(z,w) \in \mathfrak{g}_- \,. \tag{5.6}$$

It follows from (3.8) that the horizontality of $\tilde{\Phi}$ implies that

$$E^{-1}\, dE \in \mathfrak{p}_{-1} \otimes T^*(H^r \times \Delta^{n-r}).$$

Hence, writing

$$X(z,w) = \sum_{j \le -1} X_j(z,w) \;; \quad X_j \in \mathfrak{p}_j,$$

we have

$$\begin{aligned} E^{-1}\, dE \quad &= \quad \exp(-X(z,w))\, d(\exp(X(z,w))) \\ &= \quad (\mathrm{I} - X + \frac{X^2}{2} - \cdots)\,(dX_{-1} + dX_{-2} + \cdots) \\ &\equiv \quad dX_{-1} \left(\mod \left(\bigoplus_{a \le -2} \mathfrak{p}_a \right) \otimes T^*(H^r \times \Delta^{n-r} \right), \end{aligned}$$

and therefore we must have

$$E^{-1}\, dE \;=\; dX_{-1}. \tag{5.7}$$

Note that, in particular, it follows from (5.7) that $dE^{-1} \wedge dE = 0$ and equating terms according to the decomposition of $\mathfrak{g}_-$ it follows that:

$$dX_{-1} \wedge dX_{-1} = 0. \tag{5.8}$$

Theorem 5.1. *Let $(N_1, \ldots, N_r, F)$ be a nilpotent orbit and let*

$$\Gamma \colon \Delta^r \times \Delta^{n-r} \to \mathfrak{g}_-$$

be a holomorphic map with $\Gamma(0,0) = 0$.

i) *If the map*

$$\tilde{\Phi} \colon H^r \times \Delta^{n-r} \to \check{D}$$

is horizontal then it lies in D for $\mathrm{Im}(z_j) > \alpha$, where the constant α may be chosen uniformly on compact subsets of Δ^{n-r}. In other words, $\tilde{\Phi}$ is the lifting of a period map defined in a neighborhood of $0 \in \Delta^n$.

ii) *Let* $R\colon \Delta^r \times \Delta^{n-r} \in \mathfrak{p}_{-1}$ *be a holomorphic map with* $R(0,0) = 0$ *and set:*

$$X_{-1}(z,w) \;=\; \sum_{j=1}^{r} z_j\, N_j + R(t,w)\ ;\quad t_j = \exp(2\pi i z_j).$$

Then, if X_{-1} *satisfies the differential equation* (5.8)*, there exists a unique period map* Φ *defined in a neighborhood of* $0 \in \Delta^n$ *and such that* $R = \Gamma_{-1}$*.*

Proof. The first statement is Theorem 2.8 in [11] and is a consequence of the several-variables asymptotic results in [12]. The second statement is Theorem 2.7 in [9]. Its proof consists in showing that the differential equation (5.8) is the integrability condition required for finding a (unique) solution of (5.7). The result then follows from i). A proof in another context may be found in [39]. □

Theorem 5.1 means that, asymptotically, a period map consists of linear-algebraic and analytic data. The linear algebraic data is given by the nilpotent orbit or, equivalently, the polarized mixed Hodge structure. The analytic data is given by the holomorphic, $\mathfrak{p}_{-1}$-valued map Γ_{-1}.

Example 5.2. Consider a PVHS over Δ^* of Hodge structures of weight 1 on the $2n$-dimensional vector space V. We denote by Q the polarizing form. Let $\Phi\colon \Delta^* \to D/\mathrm{Sp}(V_{\mathbb{Z}}, Q)$ be the corresponding period map. The monodromy logarithm N satisfies $N^2 = 0$ and, by Example A.13 in [7], its weight filtration is:

$$W_{-1}(N) = \mathrm{Im}(N)\ ;\quad W_0(N) = \ker(N).$$

Let $F_{\lim}$ be the limiting Hodge filtration. We have a bigrading of $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = I^{0,0} \oplus I^{0,1} \oplus I^{1,0} \oplus I^{1,1}$$

defined by the mixed Hodge structure $(W(N)[-1], F_{\lim})$. The nilpotent transformation N maps $I^{1,1}$ isomorphically onto $I^{0,0}$ and vanishes on the other summands. The form $Q(\cdot, N\cdot)$ polarizes the Hodge structure on Gr_2^W and hence defines a positive definite Hermitian form on $I^{1,1}$. Similarly, we have that Q polarizes the Hodge decomposition on $V_1 := I^{1,0} \oplus I^{0,1}$. Thus, we may a basis so that

$$N = \begin{pmatrix} 0 & 0 & I_\nu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad Q = \begin{pmatrix} 0 & 0 & -I_\nu & 0 \\ 0 & 0 & 0 & -I_{n-\nu} \\ I_\nu & 0 & 0 & 0 \\ 0 & I_{n-\nu} & 0 & 0 \end{pmatrix} \tag{5.9}$$

where $\nu = \dim_{\mathbb{C}} I^{1,1}$ and the Hodge filtration $F_{\lim} = I^{1,0} \oplus I^{1,1}$ is the subspace spanned by the columns of the $2n \times n$ matrix:

$$F_{\lim} = \begin{pmatrix} 0 & 0 \\ 0 & iI_{n-\nu} \\ I_\nu & 0 \\ 0 & I_{n-\nu} \end{pmatrix} \tag{5.10}$$

The Lie algebra $\mathfrak{g}_- = \mathfrak{p}_{-1}$ and the period map can be written as:

$$\Phi(t) = \exp(\frac{\log t}{2\pi i}\, N) \cdot \exp(\Gamma(t)) \cdot F_{\lim}$$

which takes the matrix form (cf. Example 1.16 in [7]):

$$\Phi(t) = \begin{pmatrix} W(t) \\ I_n \end{pmatrix}, \tag{5.11}$$

where

$$W(t) \;=\; \begin{pmatrix} \frac{\log t}{2\pi i} I_\nu + A_{11}(t) & A_{12}(t) \\ A_{12}^T(t) & A_{22}(t) \end{pmatrix}, \tag{5.12}$$

with $A_{11}(t)$ and $A_{22}(t)$ symmetric and $A_{22}(0) = iI$; hence, $A_{22}(t)$ has positive definite imaginary part for t near zero. This computation is carried out from scratch in [25, (13.3)].

Example 5.3. We will consider a polarized variation of Hodge structure

$$\mathbb{V} \to \Delta^*$$

over the punctured disk Δ^*, of weight 3, and Hodge numbers $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$. The classifying space for such Hodge structures is the homogeneous space $D = \mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(1)\times\mathrm{U}(1)$. We will assume that the limiting mixed Hodge structure $(W(N)[-3], F_0)$ is split over $\mathbb{R}^\dagger$ and that the monodromy has maximal unipotency index, that is: $N^3 \neq 0$ while, of course, $N^4 = 0$. Hence, the bigrading defined by (W, F_0) is:

$$V_{\mathbb{C}} = I^{0,0} \oplus I^{1,1} \oplus I^{2,2} \oplus I^{3,3},$$

where each $I^{p,q}$ is one-dimensional and defined over $\mathbb{R}$. We have $N(I^{p,p}) \subset I^{p-1,p-1}$ and therefore we may choose a basis e_p of $I^{p,p}$ such that $N(e_p) = e_{p-1}$. These elements may be chosen to be real and the polarization conditions mean that the skew-symmetric polarization form Q must satisfy:

$$Q(e_3, e_0) \;=\; Q(e_2, e_1) \;=\; 1.$$

Choosing a coordinate t in Δ centered at 0, we can write the period map:

$$\Phi(t) \;=\; \exp(\frac{\log t}{2\pi i})\,\psi(t)$$

where $\psi(t)\colon \Delta \to \check{D}$ is holomorphic. Moreover, there exists a unique holomorphic map $\Gamma\colon \Delta \to \mathfrak{g}_-$, with $\Gamma(0) = 0$ and such that

$$\psi(t) \;=\; \exp(\Gamma(t)) \cdot F_0.$$

Recall also that Γ is completely determined by its (-1)-component which must be of the form:

$$\Gamma_{-1}(t) \;=\; \begin{pmatrix} 0 & a(t) & 0 & 0 \\ 0 & 0 & b(t) & 0 \\ 0 & 0 & 0 & c(t) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.13}$$

and it is easy to check that since $\Gamma(t)$ is an infinitesimal automorphism of Q, $c(t) = a(t)$.

Now, both the limiting mixed Hodge filtration and, consequently, the limiting mixed Hodge structure depend on the choice of coordinate t and we would like to

†This assumption is not necessary but is made to simplify the arguments.

understand this dependency. A change of coordinates fixing the origin must be of the form $\hat{t} = t \cdot f(t)$, with $f(0) = \lambda \neq 0$. Now, it follows from (4.12) that:

$$\hat{F}_0 = \exp\left(-\frac{\log\lambda}{2\pi i}\ N\right) \cdot F_0. \tag{5.14}$$

and

$$\hat{\psi}(\hat{t}\,) = \exp\left(-\frac{\log f(t)}{2\pi i}\ N\right) \cdot \psi(t). \tag{5.15}$$

Set $M = \exp\left(-\frac{\log\lambda}{2\pi i}\ N\right) \in \mathfrak{g}$. Then M preserves the weight filtration and maps F_0 to $\hat{F}_0$. Hence it maps the canonical bigrading $\{I^{p,q}\}$ of (W, F_0) to the canonical bigrading $\{\hat{I}^{p,q}\}$of $(W, \hat{F}_0)$ and, consequently:

$$\hat{I}^{p,q}(\mathfrak{g}) \;=\; M \cdot I^{p,q}(\mathfrak{g}) \cdot M^{-1}.$$

Hence, given (5.15) we have

$$\begin{aligned}
\hat{\psi}(\hat{t}\,) &= \exp\left(-\frac{\log f(t)}{2\pi i}\ N\right) \cdot \exp(\Gamma(t)) \cdot F_0 \\
&= \exp\left(-\frac{\log f(t)}{2\pi i}\ N\right) \cdot \exp(\Gamma(t)) \cdot M^{-1}\hat{F}_0 \\
&= \exp\left(-\frac{\log f(t)}{2\pi i}\ N\right) \cdot M^{-1} \cdot \exp(M \cdot \Gamma(t) \cdot M^{-1}) \cdot \hat{F}_0 \\
&= \exp\left(-\frac{\log(f(t)/\lambda)}{2\pi i}\ N\right) \cdot \exp(M \cdot \Gamma(t) \cdot M^{-1}) \cdot \hat{F}_0
\end{aligned}$$

It then follows by uniqueness of the lifting that

$$\exp(\hat{\Gamma}(\hat{t}\,)) = \exp\left(-\frac{\log(f(t)/\lambda)}{2\pi i}\ N\right) \cdot \exp(M \cdot \Gamma(t) \cdot M^{-1}),$$

which yields

$$\hat{\Gamma}_{-1}(\hat{t}\,) = -\frac{\log(f(t)/\lambda)}{2\pi i}\ N\ + M \cdot \Gamma_{-1}(t) \cdot M^{-1}. \tag{5.16}$$

Let us now assume, for simplicity, that $\lambda = 0$ (this amounts to a simple rescaling of the variable) then (5.17) becomes:

$$\hat{\Gamma}_{-1}(\hat{t}\,) = -\frac{\log f(t)}{2\pi i}\ N\ + \Gamma_{-1}(t). \tag{5.17}$$

Hence, given (5.13) it follows that in the coordinate

$$\hat{t} \;:=\; t\,\exp(2\pi i a(t).$$

the function $\hat{\Gamma}_{-1}(\hat{t}\,)$ takes the form

$$\hat{\Gamma}_{-1}(\hat{t}\,) \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{b}(\hat{t}\,) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.18}$$

and, consequently, the period mapping depends on the nilpotent orbit and just one analytic function $\hat{b}(\hat{t}\,)$. The coordinate $\hat{t}$ is called the *canonical coordinate* and first appeared in the work on Mirror Symmetry (cf. [37]. In fact, it was shown by Deligne that the holomorphic function $\hat{b}(\hat{t}\,)$ is related to the so-called Yukawa coupling (see [21, 1]). We refer to [9] for a full discussion of the canonical coordinates.

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