A ring structure on intersection cohomology of hypertoric varieties

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Outline

1. Hypertoric varieties
2. Minimal extension sheaves
3. Ring structure on IH
To a rational hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^d$, associate a \textit{Hypertoric variety} $\mathcal{M}_\mathcal{H}$.

- $\dim_{\mathbb{C}} \mathcal{M}_\mathcal{H} = 2d$, torus $T = (\mathbb{C}^*)^d$ acts
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- Rationally smooth $\iff$ $\mathcal{H}$ is *simple*
- Smooth $\iff$ $\mathcal{H}$ is simple and *unimodular*. 
To a rational hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^d$, associate a Hypertoric variety $M_{\mathcal{H}}$.

- $\dim_{\mathbb{C}} M_{\mathcal{H}} = 2d$, torus $T = (\mathbb{C}^*)^d$ acts
- Rationally smooth $\iff$ $\mathcal{H}$ is simple
- Smooth $\iff$ $\mathcal{H}$ is simple and unimodular
- Never compact
The toric varieties $X_P$ whose moment polyhedra are the chambers of $\mathcal{H}$ are Lagrangian subvarieties of $\mathcal{M}_H$.

If $\mathcal{M}_H$ is smooth, then every $X_P$ is smooth, and $\mathcal{M}_H = \bigcup_P T^*X_P$. 
If $\mathcal{H}$ is central, then $\mathcal{M}_H$ is affine. If $\tilde{\mathcal{H}}$ is a simplification of $\mathcal{H}$, there is a map $\mathcal{M}_{\tilde{\mathcal{H}}} \rightarrow \mathcal{M}_H$ which is an (orbifold) resolution of singularities.
If $\mathcal{H} = \{H_1, \ldots, H_n\}$ is simple, there exists a canonical ring isomorphism

$$H^*_T(\mathcal{M}_\mathcal{H}) = \mathbb{R}[e_1, \ldots, e_n]/\langle \prod_{i \in S} e_i \mid \bigcap_{i \in S} H_i = \emptyset \rangle.$$ 

This is the face ring $\mathbb{R}[\Delta_\mathcal{H}]$ of the matroid complex of $\mathcal{H}$.
Theorem (Proudfoot-Webster '04)

If $\mathcal{H}$ is central, then there is an isomorphism

$$IH^\bullet_T(\mathcal{M}_\mathcal{H}) \cong \mathbb{R}[\Delta^{bc}_\mathcal{H}]$$

of $H^\bullet_T(pt)$-modules.

$\Delta^{bc}_\mathcal{H} = "\text{broken circuit complex}"

= simplices of $\Delta_\mathcal{H}$ containing no broken circuit.

$\text{circuit} = \text{minimal non-face } C \text{ of } \Delta_\mathcal{H}$

$\text{broken circuit} = C \setminus \text{min}(C)$.

This isomorphism is not canonical. $\Delta^{bc}_\mathcal{H}$ depends on the choice of ordering of the hyperplanes, although its Betti numbers do not.
Example

Circuits: \{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}
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\[ \Delta_{bc}^{\mathcal{H}} = \{\{1, 2\}, \{1, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\} \].
Proudfoot and Speyer constructed a Cohen-Macaulay ring $R(\mathcal{H})$ which degenerates to $\mathbb{R}[\Delta^{bc}]$ for any choice of ordering:

$$R(\mathcal{H}) = \mathbb{R}[e_1, \ldots, e_n]/\left\langle \sum_{i \in C} a_i \prod_{j \in C \setminus i} e_j = 0 \right\rangle$$

where $C$ runs over all circuits, and $\sum_{i \in C} a_i v_i = 0$ is a linear dependence among the normal vectors $v_i$ to the hyperplanes $H_i$. In particular $R(\mathcal{H})$ has the same graded dimension as $\mathbb{R}[\Delta^{bc}]$.

Question

Is there a canonical identification $R(\mathcal{H}) \cong IH_\mathcal{T}(\mathcal{M}_\mathcal{H})$?
Minimal extension sheaves on fans
(Barthel-Brasselet-Fieseler-Kaup, Bressler-Lunts) give a canonical functorial computation of $IH_T$ of toric varieties. We adapt this formalism to arrangements and hypertoric varieties...
Let $L_{\mathcal{H}} =$ the lattice of flats of $\mathcal{H}$. If $\mathcal{H}$ is simple, this is just the matroid complex $\Delta_{\mathcal{H}}$.

$E \leq F$ means $E$ lies in fewer hyperplanes — $E$ is larger as a subspace of $\mathbb{R}^d$. 

![Diagram of the lattice of flats and matroid complex](image)
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For any flat $F$, define $\mathcal{A}(F) = \text{Sym}(N_F)$, where $N_F$ is the normal space to $F$ in $\mathbb{R}^d$. 

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array} \quad \begin{array}{c}
12 \\
2 \\
23 \\
24 \\
\end{array} \quad \begin{array}{c}
\mathbb{R} [x,y] \\
\mathbb{R} [x] \\
\mathbb{R} [y] \\
\mathbb{R} \\
\end{array}
\]
The quotient maps $\mathcal{A}(F) \to \mathcal{A}(E)$ when $E \leq F$ make $\mathcal{H}$ into a sheaf of graded rings on $L_{\mathcal{H}}$, with the order topology.

A sheaf $\mathcal{M}$ on $L_{\mathcal{H}}$ is an $\mathcal{A}$-module if $\mathcal{M}(F)$ is a graded $\mathcal{A}(F)$-module for each flat $F$, and the restriction maps are maps of modules.

**Definition**

An $\mathcal{A}$-module $\mathcal{L}$ is a **minimal extension sheaf** if

1. $\mathcal{L}(\emptyset) = \mathcal{A}(\emptyset) = \mathbb{R}$
2. $\mathcal{L}(F)$ is a free $\mathcal{A}(F)$-module for all $F$
3. $\mathcal{L}$ is flabby — sections extend upward
4. $\mathcal{L}$ is minimal with respect to 1, 2, and 3.
Theorem (B.-Proudfoot)

Any two minimal extension sheaves on $L_{\mathcal{H}}$ are canonically isomorphic, up to a scalar.

If $\mathcal{H}$ is rational, then there is a canonical isomorphism

$$\mathcal{L}(L_{\mathcal{H}}) \cong IH^\bullet_{\mathcal{T}}(\mathcal{M}_{\mathcal{H}}).$$
If $\mathcal{H}$ is simple, then $\mathcal{A}$ itself is a minimal extension sheaf. Its global sections are the face ring $\mathbb{R}[\Delta_{\mathcal{H}}]$. 
Another example

For the central version of our arrangement, $\mathcal{A}$ is not flabby:

\[
\begin{array}{ccc}
1 & 2 & 34 \\
\downarrow & \downarrow & \downarrow \\
1234 & 2 & 34 \\
\end{array}
\]
Another example

For the central version of our arrangement, $A$ is not flabby: sections cannot be extended to the point.

$$A = \mathbb{R}[x, y]$$
Another example

Adding an extra generator at the point, we get a flabby sheaf:

\[
A = \mathbb{R}[x, y]
\]
The sheaves $\mathcal{A}$ and $\mathcal{L}$ come from localizing equivariant cohomology and intersection cohomology

$\mathcal{M}_\mathcal{H}$ has a stratification $\bigcup_F S_F$ indexed by $L_\mathcal{H}$. The $T$-stabilizer is the same for any point $p \in S_F$, and

$$H_T^\bullet(Tp) \cong \text{Sym}((t_F)^*) = \mathcal{A}(F).$$
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$$H_T^\bullet(Tp) \cong \text{Sym}((t_F)^*) = \mathcal{A}(F).$$

$\mathcal{L}(F)$ is the equivariant IH “stalk” along $Tp$. If $p$ degenerates from a large stratum $S_E$ to a small one $S_F$, this induces a map

$$\mathcal{L}(F) \to \mathcal{L}(E)$$

which is the restriction map for the sheaf $\mathcal{L}$. 
A normal slice to the stratum $S_F$ in $\mathcal{M}_\mathcal{H}$ is isomorphic to the affine hypertoric variety $\mathcal{M}_{\mathcal{H}_F}$ defined by the localization of $\mathcal{H}$ at $F$: the central arrangement obtained by restricting to hyperplanes in $F$ and slicing.

Thus we have an isomorphism

$$\mathcal{L}(F) \cong \text{IH}_T^\bullet(\mathcal{M}_{\mathcal{H}_F}) \cong \mathbb{R}[\Delta_{\mathcal{H}_F}^{bc}] \cong R(\mathcal{H}_F).$$

For flats $E \leq F$, we can define a ring homomorphism $R(\mathcal{H}_F) \to R(\mathcal{H}_E)$ by setting the variables $e_i, i \in F \setminus E$ to zero.

With these maps, $F \mapsto R(\mathcal{H}_F)$ defines an $\mathcal{A}$-module $\mathcal{R}$. 
Theorem (B.–Proudfoot)

$\mathcal{R}$ is a minimal extension sheaf.

Corollary

If $\mathcal{H}$ is a rational central arrangement, there is a canonical isomorphism

$$R(\mathcal{H}) = \mathcal{R}(L_{\mathcal{H}}) \cong \operatorname{IH}^\bullet_T(\mathcal{M}_\mathcal{H}).$$

In particular, $\operatorname{IH}^\bullet_T(\mathcal{M}_\mathcal{H})$ carries a canonical ring structure.
Theorem (B.–Proudfoot)

$R$ is a minimal extension sheaf.

Corollary

If $\mathcal{H}$ is a rational central arrangement, there is a canonical isomorphism

$$R(\mathcal{H}) = R(L_{\mathcal{H}}) \cong IH^\bullet(\mathcal{M}_{\mathcal{H}}).$$

In particular, $IH^\bullet(\mathcal{M}_{\mathcal{H}})$ carries a canonical ring structure.

How can we understand this ring structure?
Theorem (B.–Proudfoot)

If $\mathcal{H}$ is unimodular, then the equivariant IC sheaf $\text{IC}_T(\mathcal{M}_\mathcal{H})$ can be made into a ring object in the equivariant derived category $D^b_T(\mathcal{H})$ by a multiplication map

$$\text{IC}_T(\mathcal{M}_\mathcal{H}) \otimes \text{IC}_T(\mathcal{M}_\mathcal{H}) \to \text{IC}_T(\mathcal{M}_\mathcal{H}).$$

This ring structure is unique, and it induces our ring structure on $\text{IH}^\bullet_T(\mathcal{M}_\mathcal{H})$.

This implies that the ring structure respects a number of other functorial maps besides the restrictions in the sheaf $\mathcal{R}$. For instance, restriction to the open stratum $S_{\emptyset}$ gives a ring homomorphism

$$R(\mathcal{H}) = \text{IH}^\bullet_T(\mathcal{M}_\mathcal{H}) \to H^\bullet_T(S_{\emptyset}).$$
The unimodularity hypothesis is puzzling. The sheaf $\mathcal{R}$ makes sense, gives a minimal extension sheaf, and has the “right” Betti numbers even if $\mathcal{H}$ is not unimodular, or even not rational.

But there is an isomorphism of rings:

$$H_T^\bullet(S_\emptyset) \cong \mathbb{R}[e_1, \ldots, e_n]/\langle e_1^2, \ldots, e_n^2 \rangle + \langle \sum_{i \in C} \text{sgn}(a_i) \prod_{j \in C \setminus i} e_j = 0 \rangle.$$
Why is unimodularity needed?

The unimodularity hypothesis is puzzling. The sheaf $\mathcal{R}$ makes sense, gives a minimal extension sheaf, and has the “right” Betti numbers even if $\mathcal{H}$ is not unimodular, or even not rational.

But there is an isomorphism of rings:

$$H^*_T(S_\emptyset) \cong \mathbb{R}[e_1, \ldots, e_n]/\langle e_1^2, \ldots, e_n^2 \rangle + \left\langle \sum_{i \in C} \text{sgn}(a_i) \prod_{j \in C \setminus i} e_j = 0 \right\rangle.$$

If $\mathcal{H}$ is unimodular, then $\text{sgn}(a_i) = a_i$, so this matches up with

$$R(\mathcal{H}) = \mathbb{R}[e_1, \ldots, e_n]/\left\langle \sum_{i \in C} a_i \prod_{j \in C \setminus i} e_j = 0 \right\rangle.$$
Wild speculation

Could there be some sort of “orbifold corrections” when $\mathcal{H}$ is rational but not unimodular which make a topological description of our ring structure possible?