Matroids and K-Theory

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Let $G(d,n)$ be the Grassmannian of d-planes in $\mathbb{C}^n$. Let $T = (\mathbb{C}^*)^n$, then T acts on $\mathbb{C}^n$ by rescaling coordinates and hence acts on $G(d,n)$.

Let $x \in G(d,n)$, corresponding to a linear space $L \subset \mathbb{C}^n$. For I a d-element subset of $[n] := \{1,2,\ldots,n\}$, we say $I \in \text{Matroid}(x)$ if and only if the projection $L \to \mathbb{C}^I$ is surjective. Equivalently, $I \in \text{Matroid}(x)$ if and only if $p_I(x) \neq 0$. Then $\overline{Tx}$ is the toric variety associated to the polytope

$$P_{\text{Matroid}x} := \text{ConvexHull } (e_{i_1} + \cdots + e_{i_d})_{(i_1,\ldots,i_d)\in\text{Matroid}(x)} \subset \mathbb{R}^n.$$

All the one dimensional orbits in $\overline{Tx}$ have stabilizers of the form $\{z \in T : z_i = z_j\}$. Equivalently, all the edges of $P_{\mathrm{Matroid}(x)}$ are parallel to $e_i - e_j$ for some i, $j \in [n]$.

Proposition. *(Gelfand, Goresky, MacPherson, Serganova) Let M be a nonempty collection of d-element subsets of $[n]$; let*

$$P_M := \mathrm{ConvexHull}\ (e_{i_1} + \cdots + e_{i_d})_{(i_1,\ldots,i_d)\in M}.$$

Then M is a matroid if and only if every edge of P_M is in the direction $e_i - e_j$ for some i, $j \in [n]$.

We say M is a rank d matroid on $[n]$. We call polytopes of this type *matroidal.* Note that every face of a matroidal polytope is matroidal.

The (equivariant and ordinary) cohomology and K-classes of $\overline{Tx}$ depend only on Matroid(x), and we will give explicit formulas later. Corresponding classes can be defined for matroids that don't come from points of Grassmannians. The equivariant classes are altered by permuting the elements of $[n]$, but the ordinary classes are not, *i.e.*, they are matroid isomorphism invariants.

Let M_1 and M_2 be matroids of ranks d_1 and d_2 on E_1 and E_2. Then $M_1 \oplus M_2$ is a rank $d_1 + d_2$ matroid defined on $E_1 \sqcup E_2$.

$$M_1 \oplus M_2 := M_1 \times M_2 \subset \binom{E_1}{d_1} \times \binom{E_2}{d_2} \subset \binom{E_1 \sqcup E_2}{d_1 + d_2}.$$

We have $P_{M_1 \oplus M_2} \cong P_{M_1} \times P_{M_2}$. A matroid which can not be nontrivially written as a direct sum is called connected, every matroid is uniquely expressible as a direct sum of connected matroids. If $\dim P_M = n - c$, then M has c connected components.

Combinatorialists have defined many matroid invariants. I want to think about three main examples:

- The Tutte polynomial: a two variable polynomial associated to a matroid. Special values include the β invariant, chromatic polynomial, number of bases, number of independent sets.
- Billera, Jia and Reiner's quasisymmetric function (`arXiv:math.CO/0606646`). There is a certain Hopf algebra which has, as a basis, the set of isomorphism classes of matroids. This is a combinatorial Hopf algebra in the sense of Aguair, Bergeron and Sottile and so, by their results, there is a canonical Hopf algebra morphism to the quasi-symmetric functions.
- My invariant from `arXiv:math.AG/0603551`.

All of these invariants are maps ϕ from {Isomorphism classes of matroids} to some commutative ring with the following two properties:

- ϕ multiplies in direct sums, that is to say,

$$\phi(M_1 \oplus M_2) = \phi(M_1)\phi(M_2).$$

- ϕ adds in polytope decompositions. Let $\mathring{P}_M = \bigsqcup_{F \in \mathcal{F}} \mathring{P}_F$, where all the F are matroids. Then

$$\phi(M) = \sum_{F \in \mathcal{F}} (-1)^{\dim P_M - \dim P_F} \phi(F).$$

We will see both these properties are natural from a K-theory perspective.

What is $K_\circ$?

Let X be an algebraic variety. Then $K_\circ(X)$ is the abelian group generated by coherent sheaves on X, modulo the relations $[\mathcal{A}] + [\mathcal{C}] = [\mathcal{B}]$ whenever there is a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. $K^\circ(X)$ is the subgroup generated by vector bundles. When X is smooth, as it will be in all examples we discuss, $K^\circ(X) = K_\circ(X)$.

$K^\circ(X)$ is a ring, with multiplication $[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$ for $\mathcal{E}$ and $\mathcal{F}$ vector bundles and $K_\circ(X)$ is a $K^\circ(X)$ module under the same multiplication when $\mathcal{E}$ is a vector bundle and $\mathcal{F}$ any coherent sheaf.

If $\psi : X \to Y$ is any proper map then there is a map $\psi_* : K_\circ(X) \to K_\circ(Y)$ by $[\mathcal{E}] \mapsto \sum_i (-1)^i [R^i \phi_*(\mathcal{E})]$. An especially important case is the map from a proper variety X to a point; giving a map $K_\circ(X) \to K_\circ(\mathrm{pt}) = \mathbb{Z}$. This map is the holomorphic Euler characteristic of $\mathcal{E}$, and we denote it $\chi([\mathcal{E}])$.

If $\psi : X \to Y$ is any map, then there is a map $\psi^* : K^\circ(Y) \to K^\circ(X)$ by $[\mathcal{E}] \mapsto [\psi^* \mathcal{E}]$. If $X \to Y$ is flat, the same formula gives a map $K_\circ(Y) \to K_\circ(X)$.

Equivariant versions of these theories are defined using vector bundles/sheaves equipped with group actions.
$K_\circ^T(\mathrm{pt}) = K_T^\circ(\mathrm{pt}) \cong \mathbb{Z}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$.

If $x \in G(d,n)$, then the equivariant K-class $[\mathcal{O}_{\overline{Tx}}]$ depends only on $\mathrm{Matroid}(x)$. The ordinary K-class of $[\mathcal{O}_{\overline{Tx}}]$ depends only on the isomorphism class of $\mathrm{Matroid}(x)$. We denote these classes as $\mathbb{K}^T(M)$ and $\mathbb{K}(M)$.

Let ι be the injection $G(d_1,n_1) \times G(d_2,n_2) \hookrightarrow G(d_1+d_2, n_1+n_2)$. Then

$$\iota_*(\mathbb{K}(M_1) \boxtimes \mathbb{K}(M_2)) = \mathbb{K}(M_1 \oplus M_2).$$

Let $\mathring{P}_M = \bigsqcup_{F \in \mathcal{F}} \mathring{P}_F$, where all the F are matroids. Then

$$\mathbb{K}(M) = \sum_{F \in \mathcal{F}} (-1)^{\dim P_M - \dim P_F} \mathbb{K}(F).$$

$K_T^\circ(G(d,n)) = K_\circ^T(G(d,n))$ can be described as a certain quotient of $K_T^\circ(\mathrm{pt})[u_1^{\pm 1}, \cdots u_d^{\pm 1}]^{S_d}$.

Let S be the tautological rank d bundle on $G(d,n)$, S_λ is the result of applying the λ-th Schur functor to S and s_λ the λ-th Schur symmetric function, then $s_\lambda(u)$ represents $[S_\Lambda]$. The classes $[S_\lambda]$, where λ ranges over the partitions fitting inside a $d \times (n-d)$ box, form a $K_T^\circ(\mathrm{pt})$ basis for $K_T^\circ(G(d,n))$. The linear functionals $[\mathcal{E}] \mapsto \chi([\mathcal{E}][S_\lambda])$ are a dual basis to $K_t^\circ(G(d,n))$.

The ordinary K-class is gotten by replacing each x, coming from $K_T^\circ(\mathrm{pt}) = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by a 1.

Let M be a rank d matroid on $[n]$ and, for $I \in M$, let $C_I \subset \mathbb{R}^n$ be the cone of P_M at the vertex I. (The vertex of C_I is at 0.) Let $h_I(M)(x_1, \ldots, x_n)$ be the generating function $\sum_{a \in \mathbb{Z}^n \cap C_I} x^a$, this is a power series in x_j/x_i, where $i \in I$ and $j \in [n] \setminus I$. It can be shown that $h_I(M)$ is a rational function.

With the above notations,

$$\mathbb{K}^T(M) = \sum_{I \in M} h_I(x) \prod_{j \in [n] \setminus I} \prod_{k=1}^{d} (1 - x_j u_k)$$

$$\mathbb{K}(M) = \mathbb{K}^T(M)|_{(x_1, x_2, \ldots, x_n) = (1, \ldots, 1)}.$$

$\mathbb{K}(M)$ adds in polyhedral decompositions, but it doesn't multiply in direct sums. However, if we fix e and restrict our attention to matroids with $d \geq e$, then $\mathbb{K}(M)(u_1, \ldots, u_e, 0, 0, \ldots, 0)$ does multiply in direct sums. Moreover, this is true as an identity of polynomials.

My proof is easy combinatorics. But, conceptually, it seems that this fact is related to pulling back to the partial flag variety $\mathrm{Flag}(1, 2, 3, \ldots, e-1, e, d; n)$ and then pushing down to $\mathrm{Flag}(1, 2, 3, \ldots, e; n)$.