Rigidity Theory for Matroids

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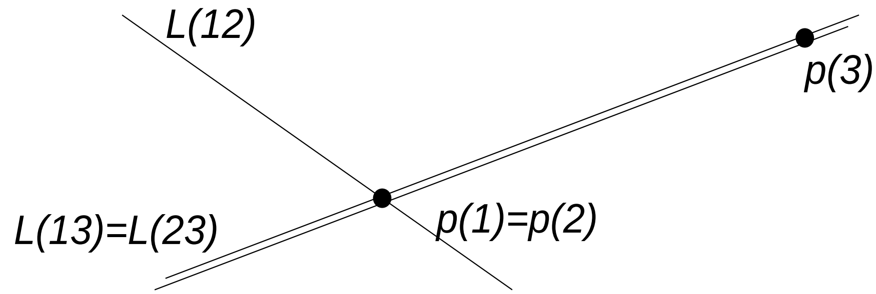
Graph Varieties

Picture space $\mathcal{X} = \mathcal{X}^d(G)$ of a graph G: a variety whose points parametrize arrangements of points and lines in $\mathbb{P}^d_{\mathbb{F}}$ that "look like" G

$G = K_3$



Combinatorics of G
(rigidity properties,
associated matroid)

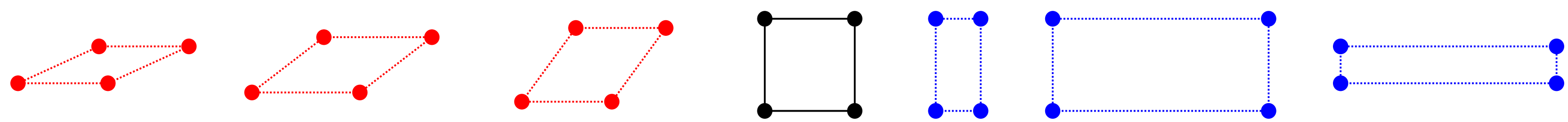
$\longleftrightarrow$

Geometry/topology of $\mathcal{X}$
(defining equations,
component structure,
homology groups,
much more)

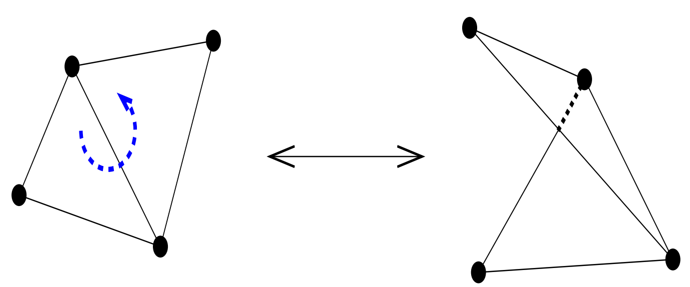
Combinatorial Rigidity Theory

A *framework* is a physical model of G built out of "joints" and "bars".

- **Pivoting framework:** the bars are fixed in length, but can pivot around their endpoints.
- **Telescoping framework:** the bars are attached to joints at fixed angles, but their lengths can change.



Question: How can we tell combinatorially whether a bar-joint framework of G is rigid? (Ambient dimension matters!)



The Rigidity Matroid

Let $G = (V, E)$, $n = |V|$, $r = |E|$, and $d \geq 2$.

d**-rigidity matrix** $R^d(G)$:

- rows indexed by E, columns indexed by dV
- column dependencies = *infinitesimal motions* preserving edge lengths
- row dependencies = *stresses* (constraints on edge lengths)

d**-rigidity matroid** $\mathcal{R} = \mathcal{R}^d(G)$ on E: represented by rows of $R^d(G)$.

- G is *d-rigid* iff $\operatorname{rank} R^d = dn - \binom{d+1}{2}$
(the only infinitesimal motions are translation and rotation)

- G is *d-rigidity-independent* iff $\mathcal{R}^d(G)$ is Boolean
(there are no constraints on the edge lengths)

The Parallel Matroid

Let $G = (V, E)$, $n = |V|$, $r = |E|$, and $d \geq 2$.

d**-parallel matrix** $P^d(G)$:

- rows indexed by E, columns indexed by dV
- column dependencies = motions preserving edge directions
- row dependencies = constraints on edge directions

d**-parallel matroid** $\mathcal{P} = \mathcal{P}^d(G)$ on E: represented by rows of $P^d(G)$.

- G is *d-parallel rigid* iff $\operatorname{rank} R^d = dn - (d+1)$

(the only infinitesimal motions are translation and scaling)

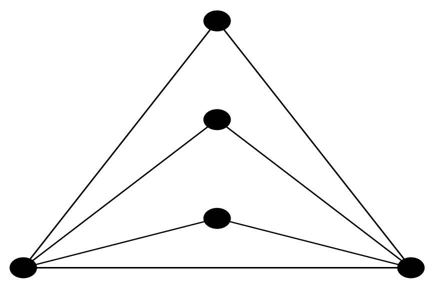
- G is *d-parallel-independent* iff $\mathcal{P}^d(G)$ is Boolean

(there are no constraints on the edge directions)

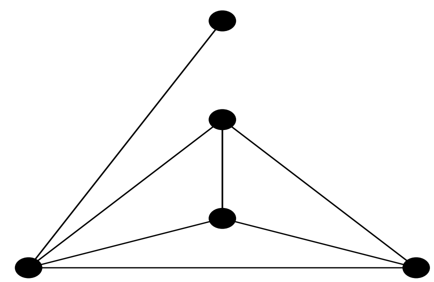
Combinatorial Rigidity in the Plane

Laman's Theorem Bases of $\mathcal{R}^2(K_n)$ = edge sets E such that

1. $|E| = 2n - 3$;
2. For every $F \subset E$, $|F| \leq 2|V(F)| - 3$.



basis of $R^2(K_5)$



not a basis

(Idea: The edges are not concentrated in any one part of the graph, as that would overdetermine some lengths and underdetermine others.)

Planar Duality Theorem $\mathcal{R}^2(G) = \mathcal{P}^2(G)$ for every graph G.

Graph Varieties and Rigidity Theory

Theorem (JLM) Let G be a graph. The following are equivalent:

(1) $\mathcal{X} = \mathcal{X}^d(G)$ is irreducible

(2) The pictures with distinct points are Zariski dense in $\mathcal{X}$

(3) $(d-1)|F| < d \cdot \mathrm{rank}(F)$ for all $\emptyset \neq F \subset E(G)$

(4) The Tutte polynomial $T_G(q, q^{d-1})$ is monic in q, of degree $|V(G)| - 1$

(5) G is d-parallel independent

Idea of proof:

Partition $\mathcal{X}$ into *cellules* indexed by set partitions of $V(G)$, and calculate dimension of each cellule.

- Putting $d = 2$ in **(3)** recovers Laman's condition
  $\Longrightarrow$ Planar Duality Theorem

Extending Rigidity Theory to Matroids

Standard proofs of Laman's and Recski's Theorems involve linear algebra and *graph-theoretic* arguments. But these results can be stated purely in terms of the underlying *matroid*.

- Can these theorems be proved matroidally?
- Are the appropriate underlying objects for combinatorial rigidity theory really *graphs*, or should we actually be studying *matroids*?
- If the latter, how can we use matroids to improve our understanding of rigidity (and vice versa)?
- What is the matroidal analogue of the picture space of a graph?
- What is the geometry behind all this?

First Approach: d-Laman Independence

Let M be a matroid on ground set E, and let $d \in (1, \infty)_{\mathbb{R}}$.

The ***d*-Laman complex of *M*** is the simplicial complex

$$\mathcal{L}^d(M) \;=\; \left\{F \subset E \mid d \cdot \operatorname{rank}(F') > |F'| \text{ for all } \emptyset \neq F' \subseteq F\right\}$$

Theorem $\mathcal{L}^d(M)$ is a matroid complex for every matroid M
$\iff d \in \mathbb{Z}$.

Theorem The following are equivalent:

(1) $T_M(q^{d-1}, q)$ is monic of degree $(d-1) \cdot \operatorname{rank}(M)$;
(2) $\mathcal{L}^d(M) = 2^E$ (generalizing Laman's condition);
(3) For every $e \in E$, the multiset $E \cup \{e\}$ can be partitioned into d disjoint independent sets (generalizing Recski's condition).

Second Approach: The Photo Space

Let M be a matroid represented over $\mathbb{F}^r$ by vectors $E = \{v_1, \ldots, v_n\}$ spanning $\mathbb{F}^r$.

Idea: Take a "d-dimensional snapshot" of M by applying a linear transformation $\phi : \mathbb{F}^r \to \mathbb{F}^d$, and record information about the directions of the vectors by requiring $\phi(v_i)$ to lie in some subspace W_i.

(Analogy: describing a picture of a graph by the direction vectors of its constituent lines.)

For integers $0 < k < d$, define the **photo space**

$$X_{k,d}(M) = \left\{ (\phi, W_1, \ldots, W_n) \;:\; \begin{array}{c} \phi \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d) \\ W_i \in \mathrm{Gr}(k, \mathbb{F}^d) \\ \phi(v_i) \in W_i \;\; \forall i \end{array} \right\}$$

M is **(k, d)-parallel independent** if the projection map

$$X_{k,d}(M) \to \mathrm{Gr}(k, \mathbb{F}^d)^n$$

is dense. (I.e., there are no mutual constraints on the W_i's.)

The **(k, d)-parallel independence complex** is defined as

$$\mathcal{P}^{k,d}(M) := \{F \subset E : \ M|_F \text{ is } (k,d)\text{-parallel independent}\}$$

Theorem The following are equivalent:

(1) M is (k, d)-parallel independent, i.e., $\mathcal{P}^{k,d} = 2^E$;

(2) $X_{k,d}(M)$ is irreducible;

(3) M is $\left(\frac{d}{d-k}\right)$-Laman independent;

(4) $d \cdot \mathrm{rank}(F) > (d-k)|F|$ for every nonempty flat F of M.

(Proof is very similar to the corresponding statement for picture spaces.)

Third Approach: The Rigidity Matroid of a Matroid

E = vectors $\{v_1, \ldots, v_n\}$ spanning $\mathbb{F}^r$
M = matroid represented by E
ϕ = $d \times r$ matrix of transcendentals (ϕ_{ij})

***d*-rigidity matroid $\mathcal{R}^d(M)$:** represented over $\mathbb{F}(\phi)$ by

$$\{v_i \otimes \phi(v_i) \mid i \in [n]\} \;\subset\; \mathbb{F}^r \otimes \mathbb{F}(\phi)^d$$

- Generalizes the graphic case: $\mathcal{R}^d(M(G)) = \mathcal{R}^d(G)$
- Can be regarded as the Jacobian of a "pseudo-distance" matrix

The Nesting Theorem For every represented matroid M,

$$\mathcal{P}^{1,d}(M) \subset \mathcal{R}^d(M) \subset \mathcal{L}^d(M) = \mathcal{P}^{d-1,d}(M).$$

Corollary $\mathcal{P}^{1,2}(M) = \mathcal{R}^2(M) = \mathcal{L}^2(M)$
(generalizing Laman's Theorem and Planar Duality Theorem)

$U_{2,4}$ and the Cross Ratio (I)

Let M be the uniform matroid $U_{2,4}$.

(Ground set E has size 4; independent sets are subsets of size ≤ 2.)

$$\mathcal{L}^d(M) \;=\; \begin{cases} U_{2,4} & \text{if } 1 < d < \frac{3}{2} \\ U_{3,4} & \text{if } \frac{3}{2} < d \leq 2 \\ U_{4,4} & \text{if } d > 2 \end{cases} \qquad (d \in \mathbb{R})$$

$$\mathcal{P}^{1,d}(M) \;=\; \begin{cases} U_{3,4} & \text{if } d = 2 \\ U_{2,4} & \text{if } d > 2 \end{cases} \qquad (d \in \mathbb{N})$$

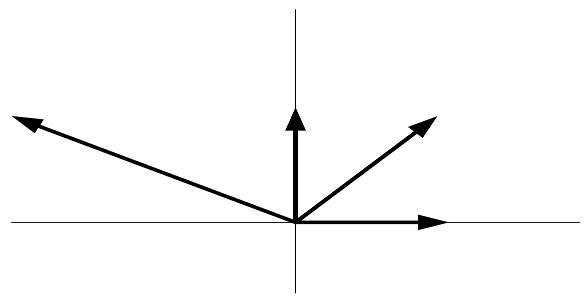
- What does this mean **geometrically**?

$U_{2,4}$ and the Cross Ratio (II)

Let $\mathbb{F}$ be a field with at least three elements, and $\mu \in \mathbb{F}^\times - \{0,1\}$

Represent $U_{2,4}$ over $\mathbb{F}$ by the vectors

$$\{v_1 = (1,0),\ v_2 = (0,1),\ v_3 = (1,1),\ v_4 = (\mu,1)\} \subset \mathbb{F}^2.$$



Recall that $\mathcal{P}^{1,2}(M) = U_{3,4}$.

We can choose $\phi : \mathbb{F}^2 \to \mathbb{F}^2$ by specifying the slopes of *three* of the $\phi(v_i)$, but not all four.

Every $\phi : \mathbb{F}^2 \to \mathbb{F}^d$ preserves the cross-ratio.

Open Questions

(1) Is $\mathcal{R}^d(M)$ independent of the particular representation?
…or at least dependent only on the choice of field?
…or at least for nice classes of matroids (e.g., graphic, uniform)?

— Yes for $d = 2$ and all M, by the Nesting Theorem.

— $\mathcal{R}^d(M)$ is well-defined up to *projective equivalence* (i.e., changing coordinates or independently scaling the v_i's)

(2) Determine the defining equations of $X_{k,d}(M)$ (e.g., the cross-ratio in the example of $U_{2,4}$).

— Probably related to decompositions of M and its minors into disjoint bases (as for the picture variety of a graph)

Open Questions

(3) For $\mathbb{F} = \mathbb{F}_q$ finite, $|X_{d,k}(M)|$ = q-binomial specialization of $T_M(x, y)$. In particular,

$$q^{dr}\,|X_{d-k,d}(M^*)| \;=\; q^{(d-k)n}\,|X_{k,d}(M)|.$$

— Is there a combinatorial explanation for this?

(∞) Characterize $\mathcal{R}^d(M)$ combinatorially???