Rigidity Theory for Matroids

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Graph Varieties

**Picture space** $\mathcal{X} = \mathcal{X}^d(G)$ of a graph $G$: a variety whose points parametrize arrangements of points and lines in $\mathbb{P}^d_F$ that “look like” $G$

$G = K_3$

**Combinatorics of $G$**
(rigidity properties, associated matroid)

**Geometry/topology of $\mathcal{X}$**
(defining equations, component structure, homology groups, much more)
Combinatorial Rigidity Theory

A framework is a physical model of $G$ built out of “joints” and “bars”.

- **Pivoting framework**: the bars are fixed in length, but can pivot around their endpoints.
- **Telescoping framework**: the bars are attached to joints at fixed angles, but their lengths can change.

**Question**: How can we tell combinatorially whether a bar-joint framework of $G$ is rigid? (Ambient dimension matters!)
The Rigidity Matroid

Let $G = (V, E)$, $n = |V|$, $r = |E|$, and $d \geq 2$.

$d$-rigidity matrix $R^d(G)$:
- rows indexed by $E$, columns indexed by $dV$
- column dependencies = *infinitesimal motions* preserving edge lengths
- row dependencies = *stresses* (constraints on edge lengths)

$d$-rigidity matroid $\mathcal{R} = \mathcal{R}^d(G)$ on $E$: represented by rows of $R^d(G)$.

- $G$ is $d$-rigid iff rank $R^d = dn - \binom{d+1}{2}$
  (the only infinitesimal motions are translation and rotation)

- $G$ is $d$-rigidity-independent iff $\mathcal{R}^d(G)$ is Boolean
  (there are no constraints on the edge lengths)
The Parallel Matroid

Let $G = (V, E)$, $n = |V|$, $r = |E|$, and $d \geq 2$.

$d$-parallel matrix $P^d(G)$:
- rows indexed by $E$, columns indexed by $dV$
- column dependencies = motions preserving edge directions
- row dependencies = constraints on edge directions

$d$-parallel matroid $\mathcal{P} = \mathcal{P}^d(G)$ on $E$: represented by rows of $P^d(G)$.

- $G$ is $d$-parallel rigid iff $\text{rank } R^d = dn - (d + 1)$
  (the only infinitesimal motions are translation and scaling)

- $G$ is $d$-parallel-independent iff $\mathcal{P}^d(G)$ is Boolean
  (there are no constraints on the edge directions)
Combinatorial Rigidity in the Plane

Laman’s Theorem  Bases of $\mathcal{R}^2(K_n) = \text{edge sets } E$ such that
1. $|E| = 2n - 3$;
2. For every $F \subset E$, $|F| \leq 2|V(F)| - 3$.

(Idea: The edges are not concentrated in any one part of the graph, as that would overdetermine some lengths and underdetermine others.)

Planar Duality Theorem  $\mathcal{R}^2(G) = \mathcal{P}^2(G)$ for every graph $G$. 
Graph Varieties and Rigidity Theory

**Theorem (JLM)** Let $G$ be a graph. The following are equivalent:

1. $\mathcal{X} = \mathcal{X}^d(G)$ is irreducible
2. The pictures with distinct points are Zariski dense in $\mathcal{X}$
3. $(d - 1)|F| < d \cdot \text{rank}(F)$ for all $\emptyset \neq F \subset E(G)$
4. The Tutte polynomial $T_G(q, q^{d-1})$ is monic in $q$, of degree $|V(G)| - 1$
5. $G$ is $d$-parallel independent

**Idea of proof:**

Partition $\mathcal{X}$ into *cellules* indexed by set partitions of $V(G)$, and calculate dimension of each cellule.

- Putting $d = 2$ in (3) recovers Laman’s condition
  $\implies$ Planar Duality Theorem
Extending Rigidity Theory to Matroids

Standard proofs of Laman’s and Recski’s Theorems involve linear algebra and graph-theoretic arguments. But these results can be stated purely in terms of the underlying matroid.

- Can these theorems be proved matroidally?
- Are the appropriate underlying objects for combinatorial rigidity theory really graphs, or should we actually be studying matroids?
- If the latter, how can we use matroids to improve our understanding of rigidity (and vice versa)?
- What is the matroidal analogue of the picture space of a graph?
- What is the geometry behind all this?
First Approach: $d$-Laman Independence

Let $M$ be a matroid on ground set $E$, and let $d \in (1, \infty)^\mathbb{R}$.

The $d$-Laman complex of $M$ is the simplicial complex

$$\mathcal{L}^d(M) = \{ F \subset E \mid d \cdot \text{rank}(F') > |F'| \text{ for all } \emptyset \neq F' \subseteq F \}$$

**Theorem** \( \mathcal{L}^d(M) \) is a matroid complex for every matroid $M$ \iff \( d \in \mathbb{Z} \).

**Theorem** The following are equivalent:

1. \( T_M(q^{d-1}, q) \) is monic of degree \((d - 1) \cdot \text{rank}(M)\);
2. \( \mathcal{L}^d(M) = 2^E \) (generalizing Laman’s condition);
3. For every \( e \in E \), the multiset \( E \cup \{e\} \) can be partitioned into \( d \) disjoint independent sets (generalizing Recski’s condition).
Second Approach: The Photo Space

Let $M$ be a matroid represented over $\mathbb{F}^r$ by vectors $E = \{v_1, \ldots, v_n\}$ spanning $\mathbb{F}^r$.

**Idea:** Take a “$d$-dimensional snapshot” of $M$ by applying a linear transformation $\phi : \mathbb{F}^r \to \mathbb{F}^d$, and record information about the directions of the vectors by requiring $\phi(v_i)$ to lie in some subspace $W_i$.

(Analogy: describing a picture of a graph by the direction vectors of its constituent lines.)

For integers $0 < k < d$, define the **photo space**

$$X_{k,d}(M) = \left\{ (\phi, W_1, \ldots, W_n) : \begin{array}{l}
\phi \in \text{Hom}_\mathbb{F}(\mathbb{F}^r, \mathbb{F}^d) \\
W_i \in \text{Gr}(k, \mathbb{F}^d) \\
\phi(v_i) \in W_i \ \forall i
\end{array} \right\}$$
$M$ is $(k, d)$-parallel independent if the projection map

$$X_{k,d}(M) \to \text{Gr}(k, F^d)^n$$

is dense. (I.e., there are no mutual constraints on the $W_i$'s.)

The $(k, d)$-parallel independence complex is defined as

$$\mathcal{P}^{k,d}(M) := \{ F \subset E : M|_F \text{ is } (k, d)\text{-parallel independent} \}$$

**Theorem** The following are equivalent:

1. $M$ is $(k, d)$-parallel independent, i.e., $\mathcal{P}^{k,d} = 2^E$;
2. $X_{k,d}(M)$ is irreducible;
3. $M$ is $(\frac{d}{d-k})$-Laman independent;
4. $d \cdot \text{rank}(F) > (d - k)|F|$ for every nonempty flat $F$ of $M$.

(Proof is very similar to the corresponding statement for picture spaces.)
Third Approach: The Rigidity Matroid of a Matroid

\[ E = \text{vectors } \{v_1, \ldots, v_n\} \text{ spanning } \mathbb{F}^r \]
\[ M = \text{matroid represented by } E \]
\[ \phi = d \times r \text{ matrix of transcendentals } (\phi_{ij}) \]

**d-rigidity matroid** \( \mathcal{R}^d(M) \): represented over \( \mathbb{F}^d(\phi) \) by

\[ \{v_i \otimes \phi(v_i) \mid i \in [n]\} \subset \mathbb{F}^r \otimes \mathbb{F}(\phi)^d \]

- Generalizes the graphic case: \( \mathcal{R}^d(M(G)) = \mathcal{R}^d(G) \)
- Can be regarded as the Jacobian of a “pseudo-distance” matrix

**The Nesting Theorem**  For every represented matroid \( M \),

\[ \mathcal{P}^{1,d}(M) \subset \mathcal{R}^d(M) \subset \mathcal{L}^d(M) = \mathcal{P}^{d-1,d}(M). \]

**Corollary**  \( \mathcal{P}^{1,2}(M) = \mathcal{R}^2(M) = \mathcal{L}^2(M) \)

(generalizing Laman’s Theorem and Planar Duality Theorem)
$U_{2,4}$ and the Cross Ratio (I)

Let $M$ be the uniform matroid $U_{2,4}$.

(Ground set $E$ has size 4; independent sets are subsets of size $\leq 2$.)

\[
\mathcal{L}^d(M) = \begin{cases} 
U_{2,4} & \text{if } 1 < d < \frac{3}{2} \\
U_{3,4} & \text{if } \frac{3}{2} < d \leq 2 \\
U_{4,4} & \text{if } d > 2
\end{cases} \quad (d \in \mathbb{R})
\]

\[
\mathcal{P}^{1,d}(M) = \begin{cases} 
U_{3,4} & \text{if } d = 2 \\
U_{2,4} & \text{if } d > 2
\end{cases} \quad (d \in \mathbb{N})
\]

- What does this mean geometrically?
$U_{2,4}$ and the Cross Ratio (II)

Let $\mathbb{F}$ be a field with at least three elements, and $\mu \in \mathbb{F}^\times - \{0, 1\}$.

Represent $U_{2,4}$ over $\mathbb{F}$ by the vectors
\[
\{v_1 = (1, 0), \ v_2 = (0, 1), \ v_3 = (1, 1), \ v_4 = (\mu, 1)\} \subset \mathbb{F}^2.
\]

Recall that $P^{1,2}(M) = U_{3,4}$.

We can choose $\phi : \mathbb{F}^2 \to \mathbb{F}^2$ by specifying the slopes of three of the $\phi(v_i)$, but not all four.

Every $\phi : \mathbb{F}^2 \to \mathbb{F}^d$ preserves the cross-ratio.
Open Questions

(1) Is $\mathcal{R}^d(M)$ independent of the particular representation?
   ...or at least dependent only on the choice of field?
   ...or at least for nice classes of matroids (e.g., graphic, uniform)?
   
   — Yes for $d = 2$ and all $M$, by the Nesting Theorem.
   
   — $\mathcal{R}^d(M)$ is well-defined up to projective equivalence
     (i.e., changing coordinates or independently scaling the $v_i$’s)

(2) Determine the defining equations of $X_{k,d}(M)$
    (e.g., the cross-ratio in the example of $U_{2,4}$).
    
    — Probably related to decompositions of $M$ and its minors into disjoint bases (as for the picture variety of a graph)
Open Questions

(3) For $\mathbb{F} = \mathbb{F}_q$ finite, $|X_{d,k}(M)| = q$-binomial specialization of $T_M(x, y)$. In particular,

$$q^{dr} |X_{d-k,d}(M^*)| = q^{(d-k)n} |X_{k,d}(M)|.$$

— Is there a combinatorial explanation for this?

(∞) Characterize $\mathcal{R}^d(M)$ combinatorially???