The toric geometry of triangulated polygons in Euclidean space

B. Howard[1] C. Manon[2] J. Millson[2]

[1] Institute for Mathematics and its Applications
University of Minnesota

[2] Department of Mathematics
University of Maryland College Park

AMS sectional conference, Storrs CT, October 2006

Motivating picture

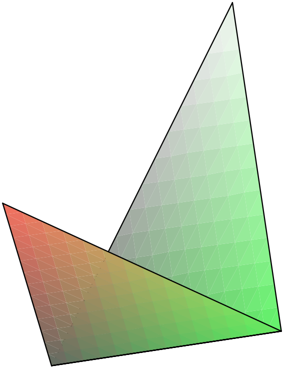


Figure: A triangulated 4-gon, bent around the diagonal.

GIT quotients, symplectic reduction

- Let **w** be an n-tuple of positive integers.
- The max. torus $T \subset SL(n)$ acts naturally on $G(2,n)$.
- This natural action may be "twisted" by **w** (by shifting the momentum map, or by the associated character $\chi_{\mathbf{w}}$ on the Plucker line bundle)
- The resulting quotient by T may be identified with the moduli space of polygonal linkages $M_{\mathbf{w}} = \{\mathbf{p} \in (\mathbb{R}^3)^n \mid \sum p_i = 0, |p_i| = w_i\}/SO(3,\mathbb{R})$.

$G(2, n)$ = "framed" n-gons.

- identify $\mathbb{R}^3 \cong \mathfrak{su}(2)^*$
- moment map $\mu : \mathbb{C}^2 \to \mathbb{R}^3$ for $SU(2)$ action is given by $(z, w) \mapsto (1/4)(|z|^2 - |w|^2, 2\Re z\overline{w}, 2\Im z\overline{w})$.
- $SU(2)$ acts diagonally on $M_{2,n}$ (2 by n matrices); a matrix is momentum level zero iff the columns C_i satisfy $\sum_i \mu(C_i) = 0$,
- Interpretation: spin-framed n-gons modulo $SU(2)$, isomorphic to $AffG(2, n) \cong M_{2,n}//SL(2)$.

The coordinate ring R_n of $G(2, n)$.

- Generators: $[i, j]$, for $1 \leq i < j \leq n$.
- Relations: $[i, l][j, k] - [i, k][j, l] + [i, j][k, l] = 0$, for $1 \leq i < j < k < l \leq n$.
- The subring of monomials generated by products $\prod_k [i_k, j_k]$ where the index i appears w_i times is the coordinate ring of $M_{\mathbf{w}}$.

Speyer-Sturmfels toric degenerations of R_n

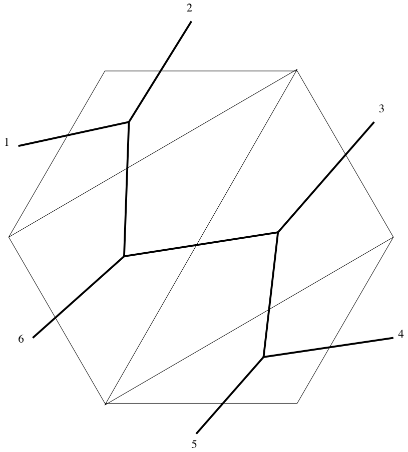


Figure: A triangulated hexagon and dual tree; $wt([i, j])$ = path length.

The toric variety $V_n^{\mathcal{T}}$

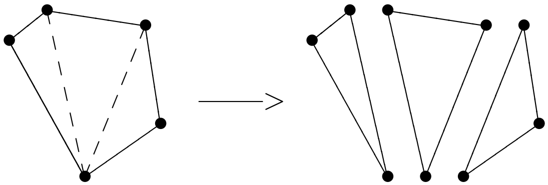
The weighting *wt* derived from the tree (or triangulation) $\mathcal{T}$ gives rise to an increasing filtration of the Grassmannian ring. The associated graded ring is toric.

Let the toric fiber be denoted $V_n^{\mathcal{T}}$, and define $M_{\mathbf{w}}^{\mathcal{T}} := V_n^{\mathcal{T}} // T$.

glueing together triangles

- The triangulation determines $n-2$ spin-frame triangles.
- Some edges are outer edges, some are internal diagonals.
- There is a natural S^1 action on each (oriented) framed edge which rotates the spin frame but leaves the primary vector fixed.
- This torus splits as $\mathbb{T} = \mathbb{T}_{edge} \times \mathbb{T}_{diag}$ and $\mathbb{T}_{diag} = \mathbb{T}^{-}_{diag} \times \mathbb{T}^{+}_{diag}$, where $\mathbb{T}^{-}_{diag}$ is the anti-diagonal action on pairs of meeting diagonals.
- The quotient by $\mathbb{T}^{-}_{diag}$ is $V_n^{\mathcal{T}}$.
- The additional quotient by $\mathbb{T}_{edge}$ is the toric fiber of $M_{\mathbf{w}}$.

Picture of the triangular decomposition



The Kamiyama-Yoshida construction

- Kapovich-Millson bending flows on $G(2, n)$ (and $M_{\mathbf{w}}$) are not well-defined where a diagonal vanishes.
- to make them well-defined - Kamiyama-Yoshida construction quotients out the “bad parts”:
- If the polygon $\mathbf{p} = \mathbf{p}_1 \vee \mathbf{p}_2$ is a wedge (vanishing diagonal), divide by $SU(2) \times SU(2)$; in general if $k - 1$ diagonals vanish, $\mathbf{p} = \mathbf{p}_1 \vee \cdots \vee \mathbf{p}_k$ then divide by $k + 1$ copies of $SU(2)$.
- There is a nice stratification (in the sense of Sjamaar-Lerman) of this space by vanishing of diagonals.

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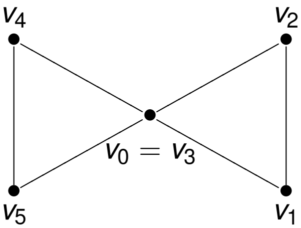
The quotient map π is not algebraic



The subspace of bowties in $M_{\mathbf{w}}$ (isomorphic to $SO(3, \mathbb{R})$) is collapsed to a point under $\pi : M_{\mathbf{w}} \to M_{\mathbf{w}}^{\mathcal{T}}$. This is for the “fan” triangulation where all diagonals initiate from the first vertex v_0. In particular, π is not a regular morphism of varieties since this subspace is odd-dimensional.

Main Theorem – Foth-Hu conjecture

The Kamiyama-Yoshida construction for $G(2, n)$ (resp. $M_{\mathbf{w}}$) coincides with the special fiber $V_n^{\mathcal{T}}$ (resp. $M_{\mathbf{w}}^{\mathcal{T}}$) of the Speyer-Sturmfels degeneration.

Bending flows vs. residual $\mathbb{T}^{+}_{diag}$ action

- The Sjamaar-Lerman stratification is by symplectic strata, and the bending flows extend to Hamiltonian flows on $V_n^{\mathcal{I}}$.
- The open set of prodigal n-gons in $M_{\mathbf{w}}$ maps symplectomorphically onto its image in $M_{\mathbf{w}}^{\mathcal{I}}$.
- On the open set of prodigal n-gons, the action of the bending flow torus on $M_{\mathbf{w}}$ (almost) coincides with the action of $\mathbb{T}^{+}_{diag}$ on $M_{\mathbf{w}}^{\mathcal{I}}$: each S^1 component of $\mathbb{T}^{+}_{diag}$ acts by spinning around the associated diagonal axis *twice*. (This comes about from the double cover $SU(2) \to SO(3, \mathbb{R})$.)

Example: $n = 4$ and $\mathbf{w} = (1, 1, 1, 1)$

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- The classical cross-ratio gives an isomorphism of $M_{\mathbf{w}}$ with $\mathbb{CP}^1$.
- Here the subspace of $M_{\mathbf{w}}$ where the diagonal vanishes is mapped onto the real line segment $[0, 1]$ of the complex plane. The unique linkage with maximal diagonal maps to ∞.
- The image of a bending-flow orbit is an ellipse with foci 0 and 1.
- Under $\pi : M_{\mathbf{w}} \to M_{\mathbf{w}}^{\mathcal{T}}$, the interval $[0, 1]$ is sent to zero, and the ellipses as above are sent to circles centered at zero.
- Thus the toric degeneration repairs the failure of the bending flows to preserve the complex structure.

Related papers

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