The toric geometry of triangulated polygons in Euclidean space

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Figure: A triangulated 4-gon, bent around the diagonal.
Let \( \mathbf{w} \) be an \( n \)-tuple of positive integers.

The maximal torus \( T \subset SL(n) \) acts naturally on \( G(2, n) \).

This natural action may be “twisted” by \( \mathbf{w} \) (by shifting the momentum map, or by the associated character \( \chi_\mathbf{w} \) on the Plucker line bundle).

The resulting quotient by \( T \) may be identified with the moduli space of polygonal linkages

\[
M_\mathbf{w} = \{ \mathbf{p} \in (\mathbb{R}^3)^n \mid \sum p_i = 0, |p_i| = w_i \}/SO(3, \mathbb{R}).
\]
G(2, n) = “framed” n-gons.

- Identify \( \mathbb{R}^3 \cong \mathfrak{su}(2)^* \)
- Moment map \( \mu : \mathbb{C}^2 \to \mathbb{R}^3 \) for \( SU(2) \) action is given by 
  \[ (z, w) \mapsto (1/4)(|z|^2 - |w|^2, 2\Re zw, 2\Im zw) \]
- \( SU(2) \) acts diagonally on \( M_{2,n} \) (2 by \( n \) matrices); a matrix is momentum level zero iff the columns \( C_i \) satisfy \( \sum_i \mu(C_i) = 0 \),
- Interpretation: spin-framed \( n \)-gons modulo \( SU(2) \), isomorphic to \( \text{Aff}G(2, n) \cong M_{2,n}/\!/SL(2) \).
The coordinate ring $R_n$ of $G(2, n)$.

- Generators: $[i, j]$, for $1 \leq i < j \leq n$.
- Relations: $[i, l][j, k] - [i, k][j, l] + [i, j][k, l] = 0$, for $1 \leq i < j < k < l \leq n$.
- The subring of monomials generated by products $\prod_k [i_k, j_k]$ where the index $i$ appears $w_i$ times is the coordinate ring of $M_w$. 
Figure: A triangulated hexagon and dual tree; $wt([i,j]) = \text{path length}$. 
The weighting $wt$ derived from the tree (or triangulation) $\mathcal{T}$ gives rise to an increasing filtration of the Grassmannian ring. The associated graded ring is toric.

Let the toric fiber be denoted $V^T_n$, and define $M^T_w := V^T_n // T$. 
The triangulation determines $n - 2$ spin-frame triangles.

Some edges are outer edges, some are internal diagonals.

There is a natural $S^1$ action on each (oriented) framed edge which rotates the spin frame but leaves the primary vector fixed.

This torus splits as $T = T_{\text{edge}} \times T_{\text{diag}}$ and $T_{\text{diag}} = T_{\text{diag}^-} \times T_{\text{diag}^+}$, where $T_{\text{diag}^-}$ is the anti-diagonal action on pairs of meeting diagonals.

The quotient by $T_{\text{diag}^-}$ is $V_T^T$.

The additional quotient by $T_{\text{edge}}$ is the toric fiber of $M_w$. 
Picture of the triangular decomposition
The Kamiyama-Yoshida construction

- Kapovich-Millson bending flows on $G(2, n)$ (and $M_w$) are not well-defined where a diagonal vanishes.

- to make them well-defined - Kamiyama-Yoshida construction quotients out the “bad parts”:
  - If the polygon $p = p_1 \lor p_2$ is a wedge (vanishing diagonal), divide by $SU(2) \times SU(2)$; in general if $k - 1$ diagonals vanish, $p = p_1 \lor \cdots \lor p_k$ then divide by $k + 1$ copies of $SU(2)$.

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The quotient map $\pi$ is not algebraic

The subspace of bowties in $M_w$ (isomorphic to $SO(3, \mathbb{R})$) is collapsed to a point under $\pi : M_w \rightarrow M_w^T$. This is for the “fan” triangulation where all diagonals initiate from the first vertex $v_0$. In particular, $\pi$ is not a regular morphism of varieties since this subspace is odd-dimensional.
The Kamiyama-Yoshida construction for $G(2, n)$ (resp. $M_w$) coincides with the special fiber $V_{n}^{T}$ (resp. $M_{w}^{T}$) of the Speyer-Sturmfels degeneration.
The Sjamaar-Lerman stratification is by symplectic strata, and the bending flows extend to Hamiltonian flows on $V^T_n$.

The open set of prodigal $n$-gons in $M_w$ maps symplectomorphically onto its image in $M^T_w$.

On the open set of prodigal $n$-gons, the action of the bending flow torus on $M_w$ (almost) coincides with the action of $T^+_\text{diag}$ on $M^T_w$: each $S^1$ component of $T^+_\text{diag}$ acts by spinning around the associated diagonal axis twice. (This comes about from the double cover $SU(2) \to SO(3, \mathbb{R})$.)
The classical cross-ratio gives an isomorphism of $M_w$ with $\mathbb{CP}^1$.

Here the subspace of $M_w$ where the diagonal vanishes is mapped onto the real line segment $[0, 1]$ of the complex plane. The unique linkage with maximal diagonal maps to $\infty$.

The image of a bending-flow orbit is an ellipse with foci 0 and 1.

Under $\pi : M_w \rightarrow M_w^T$, the interval $[0, 1]$ is sent to zero, and the ellipses as above are sent to circles centered at zero.

Thus the toric degeneration repairs the failure of the bending flows to preserve the complex structure.


A. Klyachko, *Spatial polygons and stable configurations of points on the projective line*, Algebraic geometry and its applications (Yaroslavl, 1992), 67-84.

