A Generalized Schubert Calculus for Symplectic Circle Actions

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The Set-up

(M, ω) compact symplectic manifold

S^1 acting on M with isolated fixed points M^{S^1}

Action is Hamiltonian

$$\phi : M \longrightarrow \mathbb{R}$$

$d\phi = -\omega(V, \cdot)$, V is v.f. on M generated by S^1 action.

Examples

flag manifolds
toric varieties
GKM spaces

FACT ϕ is a Morse function on M whose critical set is M^{S^1}.

Given a Morse function ϕ on M

Let g be a metric, $p \in \mathrm{crit}(\phi)$

The negative gradient flow is a map

$\psi : \mathbb{R} \times M \longrightarrow M$ such that

$$\psi(0, x) = x \quad \forall x \in M$$

$$\frac{\partial}{\partial t} \psi(t, x) = -\mathrm{grad}\, \phi(\psi(t, x)) \quad \forall t \in \mathbb{R}.$$

<u>Def'n</u> For any $p \in \mathrm{crit}(\phi)$

$$W^+(p) := \left\{ x \in M \;\middle|\; \lim_{t \to \infty} \psi(t, x) = p \right\}$$

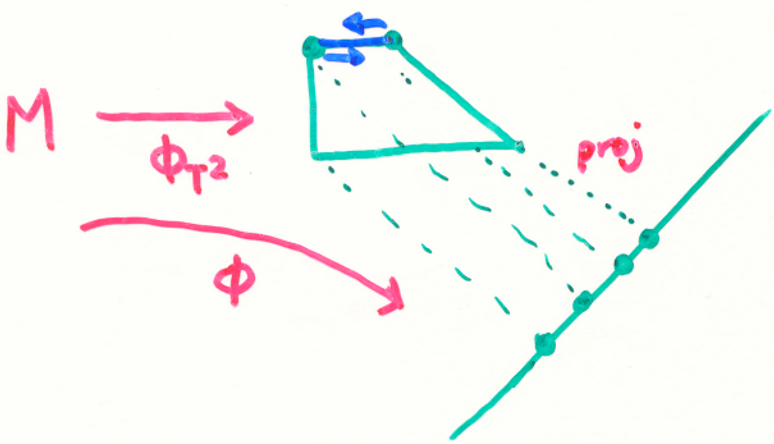
$$W^-(p) := \left\{ x \in M \;\middle|\; \lim_{t \to -\infty} \psi(t, x) = p \right\}$$

If ϕ is S^1-invariant, g S^1-invariant,

then $W^+(p)$ and $W^-(p)$ are S^1-invariant.

We say that M is Palais-Smale if $W^+(p)$ and $W^-(q)$ intersect transversally for any $p \neq q$

Ex $\mathbb{C}P^2$ blown up at a point



The intersection is not transverse.

$\Rightarrow$ not Palais-Smale.

EQUIVARIANT COHOMOLOGY

Let T = torus act on M in a Hamiltonian fashion, with M^T finite set.

- $H_T^*(M) \twoheadrightarrow H^*(M)$ surjective (& easy)

- $H_T^*(p) = \mathbb{Q}[u_1, \ldots, u_d]$ $d = \dim T$

- $H_T^*(M) \hookrightarrow H_T^*(M^T)$

$$= \bigoplus_{p \in M^T} H_T^*(p)$$ isolated fixed pts

- $H_T^*(M)$ is a module over $H_T^*(pt)$.

Lemma Let M be a compact symplectic manifold with isolated fixed points under a Hamiltonian S^1-action.

For all $p \in M^{S^1}$, there exist unique homogeneous classes $\alpha_p \in H^*_{S^1}(M; \mathbb{Z})$ such that

(1) $\alpha_p|_p = \Lambda_p^- u^i$

product of wts in negative normal bundle of p.

(2) $\alpha_p|_q = 0$ if $\mathrm{ind}(q) \leq \mathrm{ind}(p)$ or $\phi(q) \leq \phi(p)$ and $q \neq p$

Moreover, these classes generate $H^*_{S^1}(M)$ as a module over $H^*_{S^1}(pt)$.

Integrality is nontrivial
α_p is Poincaré dual in an appropriate sense to $W^+(p)$.

GOAL

Find formulas for

1. $c_{pq}^{r} \in H^{*}_{S^1}(M)$ given by

$$\alpha_p \alpha_q = \sum_r c_{pq}^{r} \alpha_r$$

2. $\alpha_p|_r$ (which determines α_p)

These problems are equivalent:
If you know c_{pq}^{r} $\forall p,q,r$, you can get $\alpha_p|_r$ $\forall p,r$ and vice-versa.

But positivity in one setting does not translate into positivity in the other.

ASSOCIATED GRAPH

Let $p, q \in M^{S^1}$, $\mathrm{ind}(q) - \mathrm{ind}(p) = 2$
$\phi(p) < \phi(q)$.

Let $L = \phi^{-1}(c)$, $\phi(p) < c < \phi(q)$.

Then each connected component of

$$W^+(p) \cap W^-(q) \cap L$$

has a <u>weight</u> associated to it.

Associated graph Γ to M

The vertices of Γ are fixed points.

Let p, q be vertices. There is an <u>oriented edge</u> from p to q for every connected component of $W^+(p) \cap W^-(q)$, if $\mathrm{ind}(q) - \mathrm{ind}(p) = 2$.

We label each vertex p by $\phi(p)$
edge by the weight $w(e)$.

Let $i(e)$ denote the initial vertex of an edge
$t(e)$ denote the terminal vertex of an edge

A path from p to q is a sequence of edges $e_1, \dots, e_k$ such that $i(e_1) = p$, $t(e_k) = q$ and $i(e_{j+1}) = t(e_j)$

<u>Thm</u> (G., Tolman) Let α_p be the canonical class associated to $p \in M^{S^1}$. Let $\text{ind}(p) = 2\ell$. For any $q \in M^{S^1}$ with $\text{ind}(q) = 2\ell + 2k$,

$$\alpha_p|_q = (-1)^k \Lambda_q^- \sum_{\substack{\text{paths from} \\ p \text{ to } q}} \prod_{j=1}^{k} \frac{\phi(t(e_j)) - \phi(i(e_j))}{w(e)} \prod_{j=1}^{k} \frac{1}{\phi(q) - \phi(i(e_j))}$$

<u>Corollary 1</u> If there is no path from p to q, then $\alpha_p|_q = 0$.

<u>Corollary 2</u> There exists a path from the minimum to any other fixed pt, and from any fixed point to the max

<u>Pf</u> If $p_0 = \min$, $\alpha_{p_0} = 1$
$\Rightarrow \alpha_{p_0}|_q = 1 \quad \forall q \Rightarrow \exists$ path $p \to q$.

A Word About Positivity

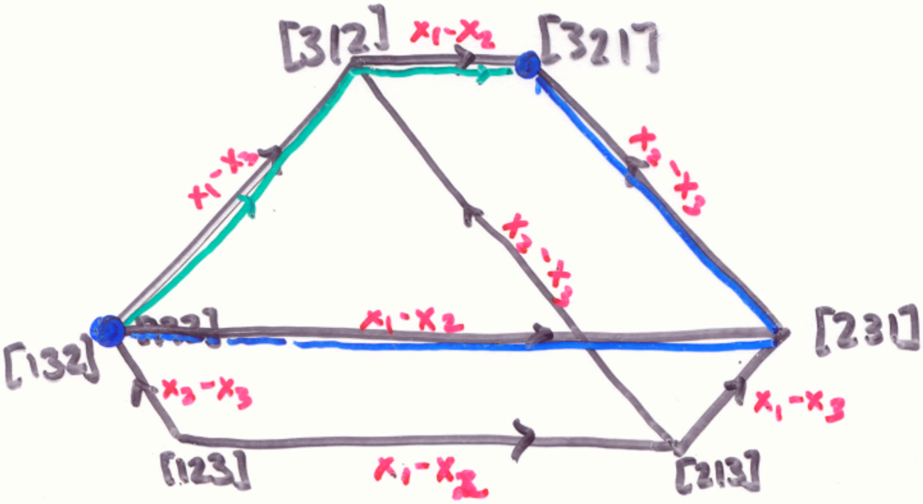
If the weights $w(e_j)$ are all positive, then

$$(-1)^k \operatorname{sgn}(\Lambda_q^-) = (-1)^k (-1)^{\ell+k} = (-1)^\ell$$
$$= \operatorname{sgn} \Lambda_p^-$$

So the sum is a positive sum.

Corollary If $w(e_j)$ are all positive, $\alpha_p|_q = 0$ iff there is no path from p to q.

Example



$Fl(\mathbb{C}^3)$

$p = [132] \quad q = [321]$

a weight at q, but not an edge weight

$$\alpha_p|_q = (-1)^2 (x_1-x_2)(x_2-x_3)(x_1-x_3)$$

$$\cdot\left(\frac{(x_2-x_3)}{x_2-x_3} \frac{(2(x_1-x_2))}{x_1-x_2} \cdot \frac{1}{(x_2-x_3)(2x_1-x_2-x_3)} + \frac{(x_1-x_2)}{(x_1-x_2)} \cdot \frac{(x_1-x_3)}{(x_1-x_3)} \frac{1}{(x_1-x_2)(2x_1-x_2-x_3)} \right)$$

$$= x_1 - x_3$$

GENERALIZED MONK

Let $p \in M^{s'}$ index 2
$q \in M^{s'}$ any index

$$\alpha_p \alpha_q = \sum c_{pq}^r \alpha_r$$
$$= c_{pq}^q \alpha_q + \sum_{\substack{\text{ind}(r) \\ = \text{ind}(q)+2}} c_{pq}^r \alpha_r$$

vanishing due to no paths + degree considerations

Then $\alpha_p|_q \alpha_p|_q = c_{pq}^q \alpha_q|_q \Rightarrow$ $c_{pq}^q = \alpha_p|_q$

$$\alpha_p \alpha_q = \alpha_p|_q \alpha_q + \sum_{\substack{\text{ind}(r) \\ = \text{ind}(q)+2}} c_{pq}^r \alpha_r .$$

Ristrict to such an r:

$$\alpha_p|_r \alpha_q|_r = \alpha_p|_q \alpha_q|_r + c_{pq}^r \alpha_r|_r \Rightarrow$$

$$\boxed{c_{pq}^r = \frac{(\alpha_p|_r - \alpha_p|_q)\alpha_q|_r}{\Lambda_r^-}}$$

GENERALIZATIONS

Let M be a cpt manifold with an S^1 action and isolated fixed points.
Suppose M has a formal moment map $\phi : M \to \mathbb{R}$, i.e. a Morse f'n whose critical points are M^{S^1}.

Ex $S^1 \curvearrowright S^4$ spinning with 2 fixed pts
ht function is a formal mom map

Suppose M has an invt Palais-Smale metric.
Then $\alpha_p \in H^*_{S^1}(M)$ still exist & are unique and are integral.

Let $p, q \in M^{S^1}$, $\phi(p) < \phi(q)$. Choose $c \in \mathbb{R}$ so that $\phi(p) < c < \phi(q)$.
Let $L := \phi^{-1}(c)$. Then

ind$(p) = 2l$
ind$(q) = 2l + 2k$

$$\alpha_p|_q = \Lambda_q^- \left(\int_{(W^+(p) \cap W^-(q) \cap L)/S^1} c_1^{k-1} \right) u^l$$

where c_1 is Chern class of $W^+(p) \cap W^-(q) \cap L \to W^+(p) \cap W^-(q) \cap L / S^1$