Quiver coefficients of Dynkin type

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Q : quiver without oriented loops

$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & 4 \arrow[d] \\ & & & 5 \end{tikzcd}$$

Vertex set: $[n] = \{1, 2, \dots, n\}$

Dimension vector: $e = (e_1, \dots, e_n)$; set $E_i = \mathbb{F}^{e_i}$, $\mathbb{F}$ field

Representation space: $V = \bigoplus_{i \to j} \mathrm{Hom}(E_i, E_j)$

Group action: $G = \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_n)$ acts on V :

$$(g_1, \dots, g_n) . (\phi_{i \to j}) = (g_j\, \phi_{i \to j}\, g_i^{-1})$$

Let $\Omega \subset V$ be any closed G-stable subvariety (e.g. orbit closure)

$\exists$ G-equivariant resolution of $\mathcal{O}_\Omega = k[V]/I(\Omega)$ by free $k[V]$-modules:

$$0 \to \mathcal{F}_r \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{O}_\Omega \to 0$$

Grothendieck class of Ω : $[\mathcal{O}_\Omega] = \sum_{i \geq 0} (-1)^i [\mathcal{F}_i] \in K_G(V)$

Stable Grothendieck polynomials

1,3	5	5,7,8
4	6, 8	

Define $\quad \mathcal{G}_\lambda(x_1, x_2, \dots) \;=\; \sum_T (-1)^{|T|-|\lambda|}\, x^T$

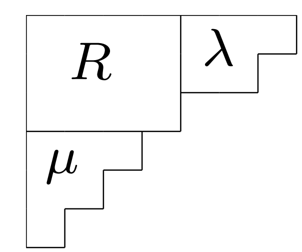
T set-valued tableaux of shape λ ; $\quad x^T = \prod x_i^{\#\text{ boxes} \ni i}$

Bialgebra of stable Grothendieck polynomials: $\quad \Gamma \;=\; \bigoplus \mathbb{Z}\,\mathcal{G}_\lambda$

Littlewood-Richardson rule (B): $\quad \mathcal{G}_\lambda \cdot \mathcal{G}_\mu \;=\; \sum_\nu c^\nu_{\lambda\mu}\, \mathcal{G}_\nu$

Coproduct (B): $\quad \Delta : \Gamma \to \Gamma \otimes \Gamma$

$\Delta(\mathcal{G}_\nu) \;=\; \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, \mathcal{G}_\lambda \otimes \mathcal{G}_\mu$; $\quad d^\nu_{\lambda\mu} = c^\rho_{R,\nu}$ where $\rho \;=$



Let U be a rational representation of G. Fix maximal torus $T \subset G$.

Write $U = L_1 \oplus \cdots \oplus L_d$, where L_i one-dim. T-representation.

Set $\;\mathcal{G}_\lambda(U) = \mathcal{G}_\lambda(1 - [L_1]^{-1}, \dots, 1 - [L_d]^{-1}, 0, 0, \dots) \;\in\; K_G(\text{point}) = K_G(V)$

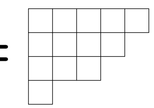
Definition: Given G-representations U_1 and U_2 , set

$$\mathcal{G}_\nu(U_1 - U_2) \;=\; \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, \mathcal{G}_\lambda(U_1) \cdot \mathcal{G}_{\mu'}(U_2^*) \quad \in K_G(V)$$

Equioriented quiver: $Q = \{1 \to 2 \to 3 \to \cdots \to n\}$

Thm (B–Fulton) Formula for $[\Omega] \in H^*_G(V)$

Thm (B) $[\mathcal{O}_\Omega] \;=\; \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_2}(E_2 - E_1) \ldots \mathcal{G}_{\mu_n}(E_n - E_{n-1}) \;\in K_G(V)$

sum over sequences $\mu = (\mu_2, \ldots, \mu_n)$ of partitions $\mu_i =$ 

Conjecture (B–Fulton, B) , **Thm** (Knutson–Miller–Shimozono, B, Miller)

Quiver coefficients have alternating signs: $(-1)^{\sum|\mu_i| - \mathrm{codim}(\Omega)}\, c_\mu(\Omega) \geq 0$.

Quiver of type A: $Q = \{1 \to 2 \leftarrow 3 \leftarrow 4 \to 5 \leftarrow 6\}$

Thm (B–Rimányi) $[\Omega] =$ positive sum of products of Schubert polynomials

Conj. (B–Rimányi) $[\mathcal{O}_\Omega] =$ alternating sum of prds. of Grothendieck polys

Quiver of Dynkin type:

Thm (Fehér–Rimányi) Linear equations determine $[\Omega]$

Thm (Knutson–Shimozono) Formula for $[\mathcal{O}_\Omega]$ using Demazure operators

Generalized quiver coefficients

For $i \in [n]$ set $M_i = \bigoplus_{j \to i} E_j$. E.g. $1 \rightrightarrows 2 \longleftarrow 3$ gives $M_2 = E_1 \oplus E_1 \oplus E_3$

Definition: Let $\Omega \subset V$ be any closed G-stable subscheme.

The **quiver coefficients** for Ω are the unique $c_\mu(\Omega) \in \mathbb{Z}$ for which

$$[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(E_1 - M_1) \dots \mathcal{G}_{\mu_n}(E_n - M_n) \quad \in K_G(V)$$

sum over sequences $\mu = (\mu_1, \dots, \mu_n)$ of partitions μ_i with $\ell(\mu_i) \leq e_i$.

Conjecture:

(1) There are finitely many non-zero quiver coefficients $c_\mu(\Omega)$ for each Ω.

(2) Cohomological quiver coefficients are non-negative, i.e.

$$\sum |\mu_i| = \mathrm{codim}(\Omega) \;\Rightarrow\; c_\mu(\Omega) \geq 0$$

(3) If Ω is irreducible with rational singularities, then the quiver coefficients for Ω have alternating signs, i.e.

$$(-1)^{\sum |\mu_i| - \mathrm{codim}(\Omega)}\, c_\mu(\Omega) \geq 0$$

Example

$Q \;=\; \{\; 1 \rightrightarrows 2 \;\}$, dimension vector $e \;=\; (3,3)$

$\Omega \;=\; \overline{G.\phi} \subset V$; $\phi \;=\; \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$

Zwara : Ω does **not** have rational singularities.

With help from Macaulay 2 :

$$
\begin{aligned}
[\mathcal{O}_\Omega] \;=\;\; & 3 \otimes \mathcal{G}_{3,1} + 4\,\mathcal{G}_1 \otimes \mathcal{G}_3 + 1 \otimes \mathcal{G}_{2,2} + 2\,\mathcal{G}_1 \otimes \mathcal{G}_{2,1} + 3\,\mathcal{G}_2 \otimes \mathcal{G}_2 + \mathcal{G}_2 \otimes \mathcal{G}_{1,1} \\
& \quad + 2\,\mathcal{G}_3 \otimes \mathcal{G}_1 + \mathcal{G}_4 \otimes 1 \\
& - 3 \otimes \mathcal{G}_{3,2} - 8\,\mathcal{G}_1 \otimes \mathcal{G}_{3,1} - 6\,\mathcal{G}_2 \otimes \mathcal{G}_3 - 2\,\mathcal{G}_1 \otimes \mathcal{G}_{2,2} - 5\,\mathcal{G}_2 \otimes \mathcal{G}_{2,1} - 4\,\mathcal{G}_3 \otimes \mathcal{G}_2 \\
& \quad - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{1,1} - 2\,\mathcal{G}_4 \otimes \mathcal{G}_1 \\
& -1 \otimes \mathcal{G}_{4,2} - 3 \otimes \mathcal{G}_{4,1,1} - 6\,\mathcal{G}_1 \otimes \mathcal{G}_{4,1} - 3\,\mathcal{G}_2 \otimes \mathcal{G}_4 - 6\,\mathcal{G}_{1,1} \otimes \mathcal{G}_4 + 4\,\mathcal{G}_1 \otimes \mathcal{G}_{3,2} \\
& \quad + 7\,\mathcal{G}_2 \otimes \mathcal{G}_{3,1} + 2\,\mathcal{G}_3 \otimes \mathcal{G}_3 + \mathcal{G}_2 \otimes \mathcal{G}_{2,2} + 4\,\mathcal{G}_3 \otimes \mathcal{G}_{2,1} + \mathcal{G}_4 \otimes \mathcal{G}_2 + \mathcal{G}_4 \otimes \mathcal{G}_{1,1} \\
& +1 \otimes \mathcal{G}_{4,3} + 5 \otimes \mathcal{G}_{4,2,1} + 10\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2} + 10\,\mathcal{G}_1 \otimes \mathcal{G}_{4,1,1} + 14\,\mathcal{G}_2 \otimes \mathcal{G}_{4,1} \\
& \quad + 15\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,1} + 4\,\mathcal{G}_3 \otimes \mathcal{G}_4 + 12\,\mathcal{G}_{2,1} \otimes \mathcal{G}_4 - \mathcal{G}_2 \otimes \mathcal{G}_{3,2} \\
& \quad - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{3,1} - \mathcal{G}_4 \otimes \mathcal{G}_{2,1} \\
& -2 \otimes \mathcal{G}_{4,3,1} - 4\,\mathcal{G}_1 \otimes \mathcal{G}_{4,3} - 1 \otimes \mathcal{G}_{4,2,2} - 16\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2,1} - 16\,\mathcal{G}_2 \otimes \mathcal{G}_{4,2} \\
& \quad - 12\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,2} - 12\,\mathcal{G}_2 \otimes \mathcal{G}_{4,1,1} - 10\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,1,1} - 10\,\mathcal{G}_3 \otimes \mathcal{G}_{4,1} \\
& \quad - 29\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,1} - \mathcal{G}_4 \otimes \mathcal{G}_4 - 7\,\mathcal{G}_{3,1} \otimes \mathcal{G}_4 - 3\,\mathcal{G}_{2,2} \otimes \mathcal{G}_4 \\
& + \cdots \\
& - \mathcal{G}_{2,1} \otimes \mathcal{G}_{4,3,2} - 2\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,3,1} - \mathcal{G}_{4,1} \otimes \mathcal{G}_{4,2,1} - 3\,\mathcal{G}_{3,2} \otimes \mathcal{G}_{4,2,1}
\end{aligned}
$$

Resolution of singularities for orbit closures of Dynkin type

Let X be a variety with a quiver $\mathcal{E}_\bullet$ of vector bundle maps:

$$\mathcal{E}_\bullet = \left\{ \begin{tikzcd} \mathcal{E}_1 \arrow[r] & \mathcal{E}_2 & \mathcal{E}_3 \arrow[l] \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{E}_4 \arrow[dl] \\ & & \mathcal{E}_5 & \end{tikzcd} \right\}$$

Fix $i \in [n]$ and $0 \le a \le e_i := \mathrm{rank}(\mathcal{E}_i)$

Set $Y = \mathrm{Gr}^a(\mathcal{E}_i) \xrightarrow{\rho} X$, tautological short exact $0 \to \mathcal{S}^{e_i-a} \to \rho^*\mathcal{E}_i \to \mathcal{Q}^a \to 0$

Set $X_{i,a} = X_{i,a}(\mathcal{E}_\bullet) = Z(\mathcal{M}_i \to \mathcal{E}_i \to \mathcal{Q}) \subset Y$, where $\mathcal{M}_i = \bigoplus_{j \to i} \mathcal{E}_j$

Note: On $X_{i,a}$ all maps $\mathcal{E}_j \to \mathcal{E}_i$ of $\mathcal{E}_\bullet$ factor through $\mathcal{S} \subset \mathcal{E}_i$

Obtain new quiver $\mathcal{E}'_\bullet$ on $X_{i,a}$ with bundles $\mathcal{E}'_j = \begin{cases} \mathcal{E}_j & \text{if } j \neq i \\ \mathcal{S} & \text{if } j = i \end{cases}$

Given $I = (i_1, \ldots, i_m)$ and $A = (a_1, \ldots, a_m)$, define

$$X_{I,A} = X_{I,A}(\mathcal{E}_\bullet) = (\cdots((X_{i_1,a_1})_{i_2,a_2})\cdots)_{i_m,a_m} \xrightarrow{\pi} X$$

Thm (Reineke) Let Q be of Dynkin type and let $\Omega \subset V$ be an orbit closure. Let $\mathcal{E}_\bullet$ be the quiver of tautological bundle maps over V , where $\mathcal{E}_i = E_i \times V$. Then $\exists$ I, A s.t. $\pi : V_{I,A}(\mathcal{E}_\bullet) \to \Omega \subset V$ is a resolution of singularities of Ω.

Thm (Reineke) Let Q be of Dynkin type and let $\Omega \subset V$ be an orbit closure. Let $\mathcal{E}_\bullet$ be the quiver of tautological bundle maps over V, where $\mathcal{E}_i = E_i \times V$. Then $\exists$ I, A s.t. $\pi : V_{I,A}(\mathcal{E}_\bullet) \to \Omega \subset V$ is a resolution of singularities of Ω.

Corollary $[\Omega] \;=\; \pi_*(\;[V_{I,A}]\;) \;\in H^*_G(V)$

Thm (Bobinski–Zwara) If Q is of Dynkin type A, or Dynkin type D in $\mathrm{char}(\mathbb{F}) = 0$, then any orbit closure $\Omega \subset V$ has rational singularities.

Corollary $[\mathcal{O}_\Omega] \;=\; \pi_*(\;[\mathcal{O}_{V_{I,A}}]\;) \;\in K_G(V)$

Main result: For any quiver Q, formula for $c_\mu(I,A) \in \mathbb{Z}$ such that

$$\pi_*(\;[\mathcal{O}_{V_{I,A}}]\;) \;=\; \sum_\mu c_\mu(I,A)\, \mathcal{G}_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1) \cdots \mathcal{G}_{\mu_n}(\mathcal{E}_n - \mathcal{M}_n) \;\in K_G(V)$$

Consequences:

(1) Only finitely many non-zero quiver coefficients for Dynkin quivers.

(2) Quiver coefficients of type A_3 have alternating signs.

Conjecture $c_\mu(I,A)$ have alternating signs.

Formula for $c_\mu(I,A)$, where $I=(i,i_2,\ldots,i_m)$, $A=(a,a_2,\ldots,a_m)$.

Construct "polynomial" $P_{I,A} = \sum_\mu c_\mu(I,A)\,\mathcal{G}_{\mu_1}\otimes\cdots\otimes\mathcal{G}_{\mu_n}$ in $\Gamma^{\otimes n}$

Induction on m : If $m=0$ then set $P_{\emptyset,\emptyset}=1\otimes\cdots\otimes 1$

Assume $P_{I',A'}\in\Gamma^{\otimes n}$ already constructed, $I'=(i_2,\ldots,i_m)$, $A'=(a_2,\ldots,a_m)$

Given $j\in[n]$, define $\psi_j:\Gamma^{\oplus n+1}\to\Gamma^{\oplus n+1}$ by

$$\psi_j(\mathcal{G}_{\mu_1}\otimes\cdots\otimes\mathcal{G}_{\mu_n}\otimes\mathcal{G}_\lambda)=\sum_{\sigma,\nu}\left(\sum_\tau d^{\mu_j}_{\sigma,\tau}\,c^\nu_{\tau,\lambda}\right)\mathcal{G}_{\mu_1}\otimes\cdots\otimes\mathcal{G}_{\mu_{j-1}}\otimes\mathcal{G}_\sigma\otimes\mathcal{G}_{\mu_{j+1}}\otimes\cdots\otimes\mathcal{G}_{\mu_n}\otimes\mathcal{G}_\nu$$

Define $\mathcal{A}:\Gamma^{\otimes n+1}\to\Gamma^{\otimes n}$ by

$$\mathcal{A}(\mathcal{G}_{\mu_1}\otimes\cdots\otimes\mathcal{G}_{\mu_n}\otimes\mathcal{G}_\nu) = \mathcal{G}_{\mu_1}\otimes\cdots\otimes\mathcal{G}_{\mu_{i-1}}\otimes\mathcal{G}_{(c)^a+\nu,\mu_i}\otimes\mathcal{G}_{\mu_{i+1}}\otimes\cdots\otimes\mathcal{G}_{\mu_n}$$

where $c=\mathrm{rank}(\mathcal{M}_i)-\mathrm{rank}(\mathcal{E}_i)+a$ and $(c)^a+\nu,\mu_i=(c+\nu_1,\ldots,c+\nu_a,\mu_i)$

Theorem Let $\{i\to j_1,\ldots,i\to j_r\}$ be the arrows of Q with tail i.

Then $P_{I,A} = \mathcal{A}(\psi_{j_1}\cdots\psi_{j_r}(\,P_{I',A'}\otimes 1\,))$