Bounded domains which are universal for minimal surfaces

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August 16, 2005

Abstract

We construct open domains in \( \mathbb{R}^3 \) which do not admit complete properly immersed minimal surfaces with an annular end. These domains can not be smooth by a recent result of Martín and Morales [7].

2000 Mathematics Subject Classification. Primary 53A10; Secondary 49Q05, 49Q10, 53C42.

Key words and phrases: Complete bounded minimal surfaces, proper minimal immersions.

1 Introduction

The main goal of this paper is to construct bounded open domains in \( \mathbb{R}^3 \) which do not contain any complete properly immersed minimal

∗ Research partially supported by MEC-FEDER Grant no. MTM2004 - 00160.
† This material is based upon work for the NSF under Award No. DMS - 0405836. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.
surfaces with at least one annular end. It is our belief that these open
domains are in fact universal according to the following definition: A
connected region of space which is open or the closure of an open set
is universal for minimal surfaces, if every complete properly immersed
minimal surface in the region is recurrent for Brownian motion. In
particular, a bounded domain is universal if and only if it contains no
complete properly immersed minimal surfaces.

**Theorem 1.** Let \( D \) be any bounded open domain in \( \mathbb{R}^3 \). Then there
exists a proper countable collection \( F \) of pairwise disjoint horizontal
simple closed curves in \( D \) such that the complementary domain \( \overline{D} = D - F \) is universal for minimal surfaces with at least one annular end.
In particular, any complete immersed minimal surface of finite genus
in \( \overline{D} \) must have an uncountable number of ends.

**Corollary 1.** There exist bounded open regions of \( \mathbb{R}^3 \) which do not ad-
mit any complete properly immersed minimal surfaces with an annular
end. In particular, these domains do not contain a complete properly
immersed minimal disk.

The construction of the domains \( \overline{D} \) that appear in the above theo-
rem are motivated by a related unpublished example of the third au-
thor. We will explain a variant of his original example at the end of
Section 2.

Interest in results like Theorem 1 dates back to an earlier question
by Calabi. Calabi asked whether or not it is possible for a complete min-
imal surface in \( \mathbb{R}^3 \) to be contained in the ball \( B = \{ x \in \mathbb{R}^3 \mid ||x|| < 1 \} \).
In [11], Nadirashvili constructed a complete minimal surface in \( B \). After
Nadirashvili negative solution to Calabi’s question, Martín and Morales
[8] proved that there exist complete properly immersed minimal disks
in \( B \). Recently [7], they improved on their original techniques and were
able to show that every bounded domain with \( C^{2,\alpha} \)-boundary admits a
complete properly immersed minimal disk whose boundary limit set is
close to a prescribed simple closed curve on the boundary of the domain.
In contrast to these existence results for complete properly immersed
minimal disks in bounded domains, Colding and Minicozzi [2] recently
proved that any complete embedded minimal surface in \( \mathbb{R}^3 \) with finite
topology is properly embedded in \( \mathbb{R}^3 \). By results of Meeks and Rosen-
berg, [10, 9], any properly embedded minimal surface of finite topology
in \( \mathbb{R}^3 \) is recurrent for Brownian motion. Hence, every domain in \( \mathbb{R}^3 \) is universal for embedded minimal surfaces of finite topology. Finally, we remark that Collin, Kusner, Meeks and Rosenberg [3] proved that any properly immersed minimal surface with boundary in a closed convex domain in \( \mathbb{R}^3 \) has full harmonic measure on its boundary.

At the end of Section 2, we give an estimate for the growth of the absolute curvature function \( |K_M| \) for any complete properly immersed minimal surface \( M \) in a smooth bounded domain \( D \subset \mathbb{R}^3 \) in terms of the distance function \( d_{\partial D} \) of \( M \) to \( \partial D \). This estimate implies the function \( |K_M|d_{\partial D}^2 \) is never bounded.

\section{Proof of Theorem 1}

Let \( D \) be an open connected bounded set of \( \mathbb{R}^3 \) and let \( \overline{D} \) denote its topological closure. Without loss of generality we may assume that \( \overline{D} \) is contained in the closed slab

\[ \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq 1\} \]

and \( \overline{D} \) contains points at heights 0 and 1.

For \( t \in (0, 1) \), let \( P_t \) denote the horizontal plane at height \( t \). Let \( C_t = D \cap P_t \), which consists of a collection \( \{C_{t,i}\}_{i \in I_t} \) of connected components, for some countable indexing set \( I_t \). For each \( t \) and for each \( i \in I_t \), choose an exhaustion of \( C_{t,i} \) by smooth compact domains \( C_{t,i,k} \), \( k \in \mathbb{N} \), and where:

- \( C_{t,i,k} \subset C_{t,i,k+1} \), \( \forall k \in \mathbb{N} \),
- \( \sup_{x \in \partial C_{t,i,k}} \text{dist}(x, \partial C_{t,i}) < \frac{1}{k}, \forall k \in \mathbb{N} \).

Finally, let \( C_t(k) \text{ def} \bigcup_{i \in I_t} C_{t,i,k} \).

Now consider the following sequence of ordered rational numbers:

\[ Q = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \ldots \right\} \]
Let $t_k$ the $k$-th rational number in $Q$. Define $\mathcal{F}$ to be the collection of boundary curves to all of the domains $\bigcup_{k \in \mathbb{N}} C_{t_k}(k)$, and define $\tilde{\mathcal{D}} \overset{\text{def}}{=} \mathcal{D} - \mathcal{F}$. We are going to see that $\tilde{\mathcal{D}}$ is open. Let $x$ be a point in $\tilde{\mathcal{D}}$ and consider $k_0 \in \mathbb{N}$ such that $1/k_0 < \text{dist}(x, \partial\mathcal{D})/2$. If we define $r_0 = \frac{1}{2} \min_{k=1, \ldots, k_0} \text{dist}(x, C_{t_k}(k))$ and $r_1 = \min \{r_0, \text{dist}(x, \partial\mathcal{D})/2\}$, then it is clear from the construction of the family $\{C_{t_k}(k)\}_{k \in \mathbb{N}}$ that $\mathbb{B}(x, r_1) \subset \tilde{\mathcal{D}}$. This proves that $\tilde{\mathcal{D}}$ is open.

Suppose that $f : M \to \tilde{\mathcal{D}}$ is a complete properly immersed minimal surface with an annular end $E$ and we will obtain a contradiction. Let $L(E)$ denote the limit set of $E$. Recall that $L(E) \overset{\text{def}}{=} f(E) - f(E)$. From the definition it is clear that $L(E)$ is a closed, connected set contained in $\partial \tilde{\mathcal{D}}$, where $\partial \tilde{\mathcal{D}} = \overline{\partial \mathcal{D}} = \partial \mathcal{D} \cup \mathcal{F}$.

Our initial goal is to prove that $x_3|_{L(E)}$ is constant, from which we will easily obtain a contradiction.

If $L(E)$ intersects one of the horizontal curves $C$ in $\mathcal{F}$, then $L(E) \subset C$ (recall that $L(E)$ is connected) and we have proved that $x_3|_{L(E)}$
is constant. So, suppose that \( p \in L(E) \subset \partial D \). If \( x_3|_{L(E)} \) is not constant, then there exists a point \( q \in L(E) \) with \( x_3(p) \neq x_3(q) \). Choose a positive rational number \( t \) which lies between \( x_3(p) \) and \( x_3(q) \). Notice that \( t \) can be represented by an infinite subsequence \( \{t_{k_1}, t_{k_2}, \ldots, t_{k_n}, \ldots\} \subset Q \). Since the plane \( P_t \) separates \( p \) and \( q \), for every subend \( E' \subset E \), \( P_t \cap E' \) is nonempty. On the other hand, the subdomains \( C_t(k_n) \) give a compact exhaustion to \( P_t \cap D \) with boundaries disjoint from \( E \). Therefore, every component of \( P_t \cap E \) is compact.

Since \( P_t \cap E \) is noncompact, then there exist a pair of disjoint simple closed curves in \( P_t \cap E \subset E \) which bound a compact domain in \( E \), since \( E \) is an annulus. But then the harmonic function \( x_3 \) restricted to this domain has an interior maximum or minimum which is impossible. This contradiction proves that \( x_3|_{L(E)} \) is constant. Let \( \alpha \) denote this constant.

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Figure 2: The subdomains \( C_t(k_n) \) give a compact exhaustion to \( P_t \cap D \) with boundaries disjoint from \( E \). Therefore, every component of \( P_t \cap E \) is compact. Since \( P_t \cap E \) is noncompact, then there exist a pair of disjoint simple closed curves in \( P_t \cap E \subset E \) which bound a compact domain in \( E \).

Our next step consists of proving that if \( x_3|_{L(E)} \) is constant, then the minimal immersion \( f : M \to \tilde{D} \) is incomplete, which is contrary to our assumptions. Indeed, the end \( E \) is conformally equivalent to \( \mathbb{D}^* = \mathbb{D} - \{0\} \), or \( A = \{z \in \mathbb{C} \mid r \leq |z| < 1\} \), for some \( 0 < r < 1 \) (see [5, Theorem IV.6.1]). The first possibility implies that \( f \) can be extended to the punctured (recall that \( f \) is a bounded harmonic map), and so \( f \) is incomplete. Then, consider a conformal parameterization of the end \( E \) by the annulus \( A = \{z \in \mathbb{C} \mid r \leq |z| < 1\} \subset \mathbb{C} \). Since \( x_3 \) is a bounded harmonic function defined on \( A \), then by Fatou’s theorem
$x_3$ has radial limit a.e. in $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Furthermore, the function $x_3$ is determined by the Poisson integral of its radial limits (see for instance [4].) Since the limit $\lim_{\rho \to 1} x_3(\rho \theta) = a$, at almost every point $\theta$ in $S^1$, then $x_3$ admits a regular extension to $\mathbb{A}$. Using Schwarz’ reflection principle for harmonic functions, $x_3$ can be extended to an open neighborhood of $A$. In particular, $\|\nabla x_3\|$ is bounded in $A$. On the other hand, as $x_2$ is also a bounded harmonic function, then a result by Bourgain [1, Theorem 2] asserts that the set

$$S = \left\{ \theta \in S^1 \mid \int_r^1 \|\nabla x_2(\rho \theta)\| \, d\rho < +\infty \right\}$$

has Hausdorff dimension 1, in particular $S$ is nonempty. Moreover, for a conformal minimal immersion it is well known [12] that $\|\nabla x_1\| \leq \|\nabla x_2\| + \|\nabla x_3\|$.

Hence, as a consequence of all these facts, if $\theta$ is a point in $S$ then

$$\int_r^1 \sqrt{\|\nabla x_1(\rho \theta)\|^2 + \|\nabla x_2(\rho \theta)\|^2 + \|\nabla x_3(\rho \theta)\|^2} \, d\rho < \infty,$$

which means that the divergent curve $f(\rho \theta)$, $\rho \in (r, 1)$, has finite length, and so $f$ is not complete. This contradiction proves the theorem.

We now explain a modification of the original unpublished example of Nadirashvili which motivated our construction of the domains $\tilde{D}$ given in Theorem 1.

Let $D$ be the open cube:

$$D = (-1, 1) \times (-1, 1) \times (-1, 1).$$

Let $F_1 = \{1\} \times [-1, 1] \times [-1, 1], F_2 = [-1, 1] \times \{1\} \times [-1, 1]$ and $F_3 = [-1, 1] \times [-1, 1] \times \{1\}$ be the three coordinate faces of $D$. Let $S_i \equiv \partial F_i$ be the related boundary square curves. As in the construction of the domains in Theorem 1, we need to define a countable proper collection $\mathcal{F}$ of planar simple closed curves in the cube $D$, so that $D - \mathcal{F}$ admits no complete properly immersed minimal surfaces with an annular end.
For a real number $\lambda$, let $\lambda S_i = \{ \lambda (x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in S_i \}$, for $i = 1, 2, 3$. Let $\mathcal{F}$ be the collection of curves

$\{ \frac{n-1}{n} S_{n \text{mod} 3} , \frac{1-n}{n} S_{n \text{mod} 3} \}_{n\in\mathbb{N}}$.

Then, a small modification of the arguments given in the proof of Theorem 1 implies that one of the coordinate functions $(x_1, x_2, x_3)$ restricted to the limit set of an annular end of a complete immersed minimal surface in $\tilde{D} \overset{\text{def}}{=} D - \mathcal{F}$ is constant. As in the proof of Theorem 1, we obtain a contradiction.

3 The asymptotic behavior of the Gaussian curvature

Finally, we prove the faster than quadratic blow up of curvature theorem.

**Theorem 2.** Let $M$ be a complete properly immersed minimal surface in a convex or smooth bounded domain, then the function $|K_M| d_\partial D^2$ is not bounded.

**Proof.** We proceed by contradiction. Assume there exists a constant $C > 0$ so that $|K_M| d_\partial D^2 \leq C^2$. Since the convex hull of a nonflat complete minimal surface with bounded curvature in $\mathbb{R}^3$ is all of $\mathbb{R}^3$ [13],
the curvature function is not bounded in $M$. Thus, take an arbitrary sequence of points $q_n \in M$ such that $|K_M(q_n)| \geq n^2$. Let $p'_n \in M \cap B_M(q_n, 1)$ be a maximum of

$$h_n = |K_M| d_M(\cdot, \partial B_M(q_n, 1))^2,$$

where $d_M$ denotes the intrinsic distance of $M$ and $B_M(q_n, 1)$ means the intrinsic ball centered at $q_n$ with radius 1.

We label $\lambda'_n = \sqrt{|K_M(p'_n)|}$. Notice that:

$$\lambda'_n \geq \lambda'_n d_M(p'_n, \partial B_M(q_n, 1)) = \sqrt{h_n(p'_n)} \geq \sqrt{h_n(q_n)} = \sqrt{|K_M(q_n)|} = n.$$

Fix $t > 0$. Notice that the sequence of extrinsic balls

$$\left\{ \lambda'_n B(p'_n, t) \right\}_{n \in \mathbb{N}}$$

converges to the ball $\mathbb{B}(t)$, where we have indentified $p'_n$ with $\vec{0}$. Similarly, we can consider $\left\{ \lambda'_n B_M(p'_n, t/\lambda'_n) \right\}_{n \in \mathbb{N}}$ as a sequence of minimal surfaces with boundary, passing through $\vec{0}$ with curvature $-1$ at the origin. From our assumption, we know that $D_n(t) = \partial D \cap \mathbb{B}(p'_n, t/\lambda'_n)$ is nonempty, for any $t > C$.

We assert that the curvature of these minimal surfaces with boundary is uniformly bounded. Indeed, pick a point $z$ in $B_M(p'_n, t/\lambda'_n)$. Then we have

$$\frac{\sqrt{|K_M(z)|}}{\lambda'_n} = \frac{\sqrt{h_n(z)}}{\lambda'_n d_M(z, \partial B_M(q_n, 1))} \leq \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z, \partial B_M(q_n, 1))} \quad (1)$$

By the triangle inequality, one has

$$d_M(p'_n, \partial B_M(q_n, 1)) \leq \frac{t}{\lambda'_n} + d_M(z, \partial B_M(q_n, 1)),$$

and so

$$\frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z, \partial B_M(q_n, 1))} \leq 1 + \frac{t}{\lambda'_n d_M(z, \partial B_M(q_n, 1))} \leq 1 + \frac{t}{\lambda'_n \left( d_M(p'_n, \partial B_M(q_n, 1)) - \frac{t}{\lambda'_n} \right)} \leq 1 + \frac{t}{n - t},$$

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which tends to 1 as $n \to \infty$.

After extracting a subsequence, it follows that $\lambda'_n B_M(p'_n, \frac{1}{\lambda'_n})$ converge smoothly to a minimal surface $M_\infty(t)$ contained in $B(t)$. Since $\lim_{n \to \infty} \lambda'_n = +\infty$, then $\lambda'_n D_n(t)$ converges either to a plane in the case that $D$ is a regular domain or to the boundary of a convex body if $D$ is a convex domain. In any case, $M_\infty(t)$ is contained in one of the halfspaces determined by the plane, or in the interior of the convex body. Note that $M_\infty = \bigcup_{t \geq C} M_\infty(t)$ is a complete nonflat minimal surface. By construction, $M_\infty$ has bounded curvature and is contained in a convex domain which is not $\mathbb{R}^3$. But this is contrary to the aforementioned result by Xavier. This contradiction proves that $|K_M| d_{\partial D}^2$ is not bounded.

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