Main Results

- **Classification** of properly embedded minimal planar domains in $\mathbb{R}^3$ (Meeks, Perez, Ros).

- **Local Removable Singularity Theorem** for minimal laminations (Meeks, Perez, Ros).

- **Solution of the Calabi-Yau problem** for arbitrary topological type (Ferrer, Martin, Meeks).

- **Proof of the Stable Limit Leaf Theorem** (Meeks, Perez, Ros).

- **Curvature estimates and sharp mean curvature bounds** for CMC foliations of 3-manifolds (Meeks, Perez, Ros).

- **Nonexistence** of non-minimal codimension one CMC foliations of $\mathbb{R}^4$ and $\mathbb{R}^5$ (Meeks, Perez, Ros).
A surface $f : M \rightarrow \mathbb{R}^3$ is minimal if:

- $M$ has **mean curvature** $= 0$.
- Small pieces have **least area**.
- Small pieces have **least energy**.
- Small pieces occur as **soap films**.
- Coordinate functions are **harmonic**.
- Conformal Gauss map $G : M \rightarrow S^2 = \mathbb{C} \cup \{\infty\}$.

**Meromorphic Gauss map**
Meromorphic Gauss map

\[ f : S^2 \rightarrow \mathbb{R}^3 \]

\[ G = \text{Gauss map} \]

\[ G(p) \in \mathbb{C} \]

\[ \text{COMPLEX PLANE} \]
Suppose $f : M \subset \mathbb{R}^3$ is minimal, 

$$g : M \rightarrow \mathbb{C} \cup \{\infty\},$$

is the meromorphic Gauss map, 

$$dh = dx_3 + i \ast dx_3,$$

is the holomorphic height differential. Then 

$$f(p) = \text{Re} \int^{p} \frac{1}{2} \left[ \frac{1}{g} - g, \frac{i}{2} \left( \frac{1}{g} + g \right), 1 \right] dh.$$
1776 Meusnier - the Helicoid

\[ M = C \]
\[ dh = dz = dx + i \, dy \]
\[ g(z) = e^{iz} \]

Helicoid
\[ M = C - \{(0, 0)\} \]

\[ dh = \frac{1}{z} dz \]

\[ g(z) = z \]
1860 Riemann’s discovery!

I am foliated by circles
The family $\mathcal{R}_t$ of Riemann minimal examples

Riemann's Infinite Staircase

- Catenoid Soap Film
- Perturbed Soap Film

Shifted wire
Cylindrical parametrization of a Riemann minimal example
Cylindrical parametrization of a Riemann minimal example
Conformal compactification of a Riemann minimal example

Top End = North Pole

Bottom End = South Pole
The moduli space of genus-zero examples
Riemann minimal examples near helicoid limits

singular lines

$S_1$ $S_2$

limit foliation by planes
Classification of infinite topology $g = 0$ examples

**Theorem (Meeks, Perez and Ros)**

A **PEMS** in $\mathbb{R}^3$ with genus zero and infinite topology is a Riemann minimal example.

We now outline the main steps of the proof of this theorem.

Throughout this outline,

$M \subset \mathbb{R}^3$ denotes a **PEMS** with genus zero and infinite topology.
Step 1: Control the topology of $M$

**Theorem (Frohman-Meeks, C-K-M-R)**

Let $\Delta \subset \mathbb{R}^3$ be a PEMS with an infinite set of ends $\mathcal{E}$. After a rotation of $\Delta$,

- $\mathcal{E}$ has a natural linear ordering by relative heights of the ends over the $xy$-plane;
- $\Delta$ has one or two limit ends, each of which must be a top or bottom end in the ordering.

**Theorem (Meeks, Perez, Ros)**

The surface $M$ has two limit ends.

**Idea of the proof** $M$ has 2 limit ends. One studies the possible singular minimal lamination limits of homothetic shrinkings of $M$ to obtain a contradiction if $M$ has only one limit end.
A proper $g = 0$ surface with uncountable # of ends

$S^2$ – Cantor set
Step 2: Understand the geometry of $M$

$M$ can be parametrized **conformally** as $f : (S^1 \times \mathbb{R}) - \mathcal{E} \to \mathbb{R}^3$ with $f_3(\theta, t) = t$ so that:

- The **middle ends** $\mathcal{E} = \{(\theta_n, t_n)\}_{n \in \mathbb{Z}}$ are **planar**.
- $M$ has **bounded curvature**, **uniform local area estimates** and is **quasiperiodic**.
- For each $t$, consider the plane curve $\gamma_t(\theta) = f(\theta, t)$ with speed $\lambda = \lambda_t(\theta) = |\gamma'_t(\theta)|$ and geodesic curvature $\kappa = \kappa_t(\theta)$. Then the **Shiffman function** $S_M = \lambda \frac{\partial \kappa}{\partial \theta}$ extends to a **bounded analytic function** on $S^1 \times \mathbb{R}$.
- $S_M$ is a **Jacobi function** when considered to be defined on $M$. $(\Delta - 2K_M) S_M = 0$. 
Step 3: Prove the Shiffman function $S_M$ is integrable

$S_M$ is **integrable** in the following sense. There exists a family $M_t$ of examples with $M_0 = M$ such that the normal variational vector field to each $M_t$ corresponds to $S_{M_t}$.

The proof of integrability of $S_M$ depends on:

- $(\Delta - 2K_M)$ has finite dimensional bounded kernel;

- $S_M$ viewed as an infinitesimal variation of Weierstrass data defined on $\mathbb{C}$, can be formulated by the KdV evolution equation.

**KdV theory completes proof of integrability.**
The Korteweg-de Vries equation (KdV)

\[ \dot{g}_S = \frac{i}{2} \left( g''' - 3 \frac{g'g''}{g} + \frac{3}{2} \left(\frac{g'}{g^2}\right)^3 \right) \in T_g \mathcal{W} \quad \text{(Shiffman)} \]

**Question:** Can we integrate \( \dot{g}_S \)? (This solves the problem)

\[ \dot{g}_S \xrightarrow{x=g'/g} (\text{mKdV}) \quad \text{Miura transf} \quad (\text{KdV}) \]

\[ \dot{x} = \frac{i}{2} \left( x''' - \frac{3}{2} x^2 x' \right) \xrightarrow{u=ax'+bx^2} \dot{u} = -u''' - 6uu' \]

\[ u = -\frac{3(g')^2}{4g^2} + \frac{g''}{2g} \]

**KdV hierarchy** (infinitesimal deformations of \( u \))

\[
\begin{align*}
\frac{\partial u}{\partial t_0} &= -u' \\
\frac{\partial u}{\partial t_1} &= -u''' - 6uu' \\
\frac{\partial u}{\partial t_2} &= -u^{(5)} - 10uu''' - 20u' u'' - 30u^2 u' \\
&\vdots
\end{align*}
\]

\( u \) algebro-geometric \( \iff \exists n, \frac{\partial u}{\partial t_n} \in \text{Span}\{\frac{\partial u}{\partial t_0}, \ldots, \frac{\partial u}{\partial t_{n-1}}\} \)

All flows commute: \( \frac{\partial}{\partial t_n} \frac{\partial u}{\partial t_m} = \frac{\partial}{\partial t_m} \frac{\partial u}{\partial t_n} \)
Step 4: Show $S_M = 0$

The property that $S_M = 0$ is equivalent to the property that $M$ is foliated by circles and lines in horizontal planes.

**Theorem (Riemann 1860)**

*If $M$ is foliated by circles and lines in horizontal planes, then $M$ is a Riemann minimal example.*

**Holomorphic integrability** of $S_M$, together with the **compactness** of the moduli space of embedded examples of fixed flux, forces $S_M$ to be **linear**, which requires the analytic data defining $M$ to be **periodic**. In 1997, we proved that $S_M = 0$ for periodic examples. Hence, $M$ is a **Riemann minimal example**.
Examples of foliations and laminations in the plane

\( \mathcal{F} = \) integral curves of a vector field.

\( \mathcal{L} = \) union of \( S^1 \) and green and red spirals
Theorem (Geodesic lamination closure theorem)

If $\Delta$ is a Riemannian surface and $\Gamma \subset \Delta$ is a complete embedded geodesic, then the closure $\bar{\Gamma}$ is a geodesic lamination of $\Delta$. 
Conjecture (Gulliver, Lawson)

If $\mathbf{M} \subset \mathbf{B} - \{(0,0,0)\}$ is a smooth properly embedded minimal surface with $\partial \mathbf{M} \subset \partial \mathbf{B}$ and $\overline{\mathbf{M}} = \mathbf{M} \cup \{(0,0,0)\}$, then $\overline{\mathbf{M}}$ is a smooth compact minimal surface.
Theorem (Meeks, Perez, Ros)

Let $S \subset N$ be a closed countable set in a 3-manifold $N$ and let $\mathcal{L}$ be a minimal lamination of $N - S$. If in some small neighborhood of every isolated point $p$ of $S$, $|K_{\mathcal{L}}(x)| \leq \frac{C_p}{d^2(x,p)}$, then:

- $\mathcal{L}$ extends across $S$ to a minimal lamination $\overline{\mathcal{L}}$ of $N$.
- The sublamination $\text{Lim}(\overline{\mathcal{L}}) \subset \overline{\mathcal{L}}$ of limit leaves consists of stable minimal surfaces.
Theorem (Meeks, Perez, Ros)

Let $M \subset N$ be a complete embedded finite topology minimal surface in a complete Riemannian 3-manifold. If $\overline{M}$ is not a minimal lamination with $M$ as a leaf, then the following hold:

- $\mathcal{L} = (\overline{M} - M)$ is a minimal lamination of $N$ with leaves whose two-sided covers are stable.
- $M$ is proper in $N - \mathcal{L}$.
- If $N$ is compact, then $\mathcal{L}$ contains a leaf which is an embedded sphere or projective plane.
Theorem (Old Conjecture, Meeks, Perez, Ros)

A complete embedded minimal surface of finite topology in the $3$-sphere $S^3 \subset \mathbb{R}^4$ is compact.

Proof.

Since $M$ is noncompact, then $\overline{M}$ is a minimal lamination with a limit leaf $L$ or $\overline{M} - M$ is a minimal lamination with a leaf $L$ whose two-sided cover is stable. By the Stable Limit Leaf Theorem, in either case the two-sided cover of $L$ is stable. But complete stable two-sided minimal surfaces do not exist in positive Ricci curvature $3$-manifolds!
Colding and Minicozzi proved the next result in the case of finite topology.

Theorem (Meeks, Perez, Ros)

If $M \subset \mathbb{R}^3$ be a complete, connected embedded minimal surface with finite genus, a countable number of ends and compact boundary, then $M$ is properly embedded in $\mathbb{R}^3$.

In particular, if $\Sigma \subset \mathbb{R}^3$ is a complete embedded bounded minimal surface, then every end of $\Sigma$ has infinite genus or is a genus zero limit end.
Theorem (Embedded Topological Obstruction, Ferrer, Martin, Meeks)

If $M$ is a nonorientable surface and has an infinite number of nonorientable ends, then $M$ cannot properly embed in any smooth bounded domain of $\mathbb{R}^3$.

Theorem (Immersed Topological Obstruction, Martin, Meeks, Nadirashvili)

There exist bounded domains $D \subset \mathbb{R}^3$ which do not admit any complete, properly immersed minimal surfaces with at least one annular end.
Let $\mathcal{M}$ be open surface.

1. There exists a complete proper minimal embedding of $\mathcal{M}$ in every smooth bounded domain $D \subset \mathbb{R}^3$ iff $\mathcal{M}$ is orientable and every end has infinite genus.

2. There exists a complete proper minimal embedding of $\mathcal{M}$ in some smooth bounded domain $D \subset \mathbb{R}^3$ iff every end of $\mathcal{M}$ has infinite genus and $\mathcal{M}$ has a finite number of nonorientable ends.

3. There exists a complete proper minimal embedding of $\mathcal{M}$ in some particular non-smooth bounded domain $D \subset \mathbb{R}^3$ iff every end of $\mathcal{M}$ has infinite genus.
Theorem (Solution of the Calabi-Yau Problem for Arbitrary Topology, Ferrer, Martin, Meeks)

Let $D$ be a domain which is convex (possibly $D = \mathbb{R}^3$) or smooth and bounded. Given any open surface $M$, there exists a complete proper minimal immersion $f : M \to D$, such that the limit sets of distinct ends of $M$ are disjoint.

This result and its proof represent the first key point in my approach with Martin and Nadirashvili to solve the existence implication in the Embedded Calabi-Yau Conjecture, including the nonorientable case.
Universal domain for the Calabi-Yau problem?

$D = \text{bounded domain, smooth except at } p_\infty$. Ferrer, Martin and Meeks conjecture every open surface with only infinite genus ends properly embeds as a complete minimal surface in $D$. 
Theorem (Stable Limit Leaf Theorem, Meeks, Perez, Ros)

The limit leaves of a codimension one $H$-lamination $L$ of a Riemannian manifold $N$ are stable.

Proof.

Assume: Dimension $(N) = 3$.

First step: Interpolation result.

Below $D(p, r)$ is a disk in a limit leaf $L$ and the blue arcs represent graphical disks in leaves converging to $L$.

The interpolating graphs $f_t$ between the $H$-graphs of $f_{t_{\alpha}}$, $f_{s_{\alpha}}$ satisfy

$$\lim_{t \to 0^+} \frac{H_t(q) - H}{t} = 0 \quad \text{for all } q \in D(p, r).$$
The interpolating graphs $q \mapsto \exp_q(f_t(q)\eta(q))$, $t \in [t_\alpha, s_\alpha]$, where

\[
f_t = f_{t_\alpha} + (t - t_\alpha) \frac{f_{s_\alpha} - f_{t_\alpha}}{s_\alpha - t_\alpha} = t \left[ \frac{t_\alpha}{t} \cdot \frac{f_{t_\alpha}}{t_\alpha} + \left(1 - \frac{t_\alpha}{t}\right) \cdot \frac{f_{s_\alpha} - f_{t_\alpha}}{s_\alpha - t_\alpha} \right],
\]
satisfy

\[
\lim_{t \to 0^+} \frac{H_t(q) - H}{t} = 0 \quad \text{for all } q \in D(p, r).
\]
Assume: $H = 0$ and $\Delta \subset L = \text{unstable}$ smooth compact subdomain. Let $\Delta(s)$ be surfaces whose mean curvature increases to first order near $\Delta$ and foliate the shaded region $\Omega(t)$ between $\Delta$ and $\Delta(t)$. Let $V$ be the unit normal field to this foliation. Let $W$ be the unit normal field to the red interpolated foliation containing $\mathcal{L}$. Note $\text{Div}(V) \leq \text{Div}(W)$ in $\Omega(t)$. But the flux of $V$ across $\partial \Omega(t)$ is greater than the flux of $W$ across the same boundary. The divergence theorem gives a contradiction.
Theorem (Curvature Estimates, Meeks, Perez, Ros)

Given $K \geq 0$, there exists $C_K \geq 0$ such that whenever $N$ is a complete 3-manifold with absolute curvature bounded by $K$ and $\mathcal{F}$ is a CMC foliation of $N$, then $|A|_\mathcal{F} \leq C_K$. Here $|A|_\mathcal{F}$ is the norm of the second fundamental form of the leaves of $\mathcal{F}$.

Corollary (Meeks)

A CMC foliation of $\mathbb{R}^3$ is a foliation by parallel planes.

Corollary (Mean Curvature Bounds, Meeks, Perez, Ros)

If $N$ is a complete 3-manifold with bounded absolute sectional curvature, then there is a uniform bound on the mean curvature of the leaves of any CMC foliation of $N$. 
Proof.

After scaling and lifting to the universal cover, assume $K \leq 1$. If the theorem fails, there exist CMC foliations $\mathcal{F}_n$ of $N$ and a sequence of "blow-up" points $p_n \in N$ on leaves $L_n$, where $\lambda_n = |A|_{L_n} \geq n$. The foliated metrically scaled balls $\lambda_n B(p_n, 1)$ converge to a "singular CMC foliation" $\mathcal{Z} = \{\Sigma_\alpha\}_\alpha$ of $R^3$ such that:

- $|A|_\mathcal{Z} \leq 1$.
- The leaf $\Sigma$ passing through the origin is nonflat.
- $\mathcal{Z}$ is not a minimal foliation.

Since $|A|_\mathcal{Z} \leq 1$, after translations of $\mathcal{Z}$, we obtain another limit singular CMC foliation of $R^3$ with a leaf passing through the origin having maximal nonzero mean curvature. But this leaf is then a stable sphere which is impossible.
Theorem (Meeks, Perez, Ros)

Suppose that $\mathbb{N}$ is $\mathbb{R}^3$ equipped with a complete homogeneously regular metric satisfying: the scalar curvature of $\mathbb{N}$ is bounded from below by a nonpositive constant $-C$. Suppose $\mathcal{F}$ is a CMC foliation of $\mathbb{N}$. Then:

- The mean curvature $H$ of any leaf of $\mathcal{F}$ satisfies $H^2 \leq C$.
- Leaves of $\mathcal{F}$ with $|H| = \sqrt{C}$ are stable, have at most quadratic area growth and are asymptotically umbilic.
- If $C \geq 0$, then $\mathcal{F}$ is a minimal foliation.

Corollary (Meeks, Perez, Ros)

The leaves of a codimension one CMC foliation of $\mathbb{H}^3$ have absolute mean curvature at most 1 and each leaf with absolute mean curvature 1 is a horosphere.
Theorem (Meeks, Perez, Ros)

Suppose that $N$ is a complete homogeneously regular manifold of dimension at most 5 and $\mathcal{F}$ is a codimension one CMC foliation of $N$. There exists a bound on the absolute mean curvature $H$ of any leaf of $\mathcal{F}$ depending only on an upper bound of the absolute sectional curvature of $N$.

Ingredients of the proof:

- Nonexistence of stable $H$-hypersurfaces in $\mathbb{R}^3$ (C, E-N-R)
- Stable minimal hypersurfaces in $\mathbb{R}^5$ with Euclidean volume growth are hyperplanes (Schoen, Simon, Yau)

- Stable Limit Leaf Theorem

Corollary (Meeks, Perez, Ros)

A codimension one CMC foliation of $\mathbb{R}^n$, $n \leq 5$, is minimal.