Calculus on the complex plane $\mathbb{C}$

Let $z = x + iy$ be the complex variable on the complex plane $\mathbb{C} = \mathbb{R} \times i\mathbb{R}$ where $i = \sqrt{-1}$.

**Definition**

- A function $f: \mathbb{C} \to \mathbb{C}$ is **holomorphic** if it is complex differentiable, i.e., for each $z \in \mathbb{C}$,

  $$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

  exists.

- When $f$ is non-constant, this is equivalent to the property: Except for isolated points, if $c_1, c_2$ are any two orthogonal curves passing through a point $p \in \mathbb{C}$, then their image curves $f \circ c_1, f \circ c_2$ are orthogonal curves at $f(p)$.

- This property is also equivalent to the property that the function $f$ is angle preserving (**conformal**) except at isolated points, when it is not constant.
Definition

- A function $f : M_1 \to M_2$ between two surfaces is called \textbf{holomorphic} if it is angle preserving except at isolated points, when it is not constant. It is called \textbf{meromorphic} if $M_2 = S^2$ is the unit sphere in $\mathbb{R}^3$.
- A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called \textbf{harmonic} if $f$ satisfies the \textbf{mean value property}, i.e., the value of $f(p)$ at any point $p \in \mathbb{R}^2$, is equal to the average value of the function on every circle centered at $p$.
- Also $f : \mathbb{R}^2 \to \mathbb{R}$ is \textbf{harmonic} if its Laplacian vanishes, $\Delta f = 0$.

Example

If $T$ is a temperature function in equilibrium on a domain $D$ in the plane, then $T$ is a harmonic function.

Theorem

A harmonic function $f : M \to \mathbb{R}$ on a surface is the real part of some holomorphic function $F : M \to \mathbb{C} = \mathbb{R} \times i\mathbb{R}$. 
Definition of minimal surface

A surface $f : M \to R^3$ is minimal if:

- $M$ has **mean curvature** $= 0$.
- Small pieces have **least area**.
- Small pieces have **least energy**.
- Small pieces occur as **soap films**.
- Coordinate functions are **harmonic**.
- Conformal Gauss map $G : M \to S^2 = C \cup \{\infty\}$.
- **Meromorphic Gauss map**
Meromorphic Gauss map

\[ S^2 \rightarrow f \rightarrow \mathbb{R}^3 \]

\[ G = \text{Gauss map} \]

\[ G(p) \in \mathbb{C} \]

\[ \text{COMPLEX PLANE} \]
Suppose $f : M \subset \mathbb{R}^3$ is minimal,

$$g : M \to \mathbb{C} \cup \{\infty\},$$

is the meromorphic Gauss map,

$$dh = dx_3 + i \ast dx_3,$$

is the holomorphic height differential. Then

$$f(p) = \text{Re} \left[ \int^p \frac{1}{2} \left[ \frac{1}{g} - g, \frac{i}{2} \left( \frac{1}{g} + g \right) \right], 1 \right] dh.$$
\[ M = C \]

\[ dh = dz = dx + i \, dy \]

\[ g(z) = e^{iz} \]
$M = C - \{(0, 0)\}$

$dh = \frac{1}{z}dz$

$g(z) = z$
Key Properties:

- In 1741, **Euler** discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the $x_3$-axis, then one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves.

- In 1776, **Meusnier** verified that the catenoid has zero mean curvature.

- This surface has genus zero, two ends and total curvature $-4\pi$. 
Key Properties:

- Together with the plane, the catenoid is the only minimal surface of revolution (*Euler* and *Bonnet*).
- It is the unique complete, embedded minimal surface with genus zero, finite topology and more than one end (*López* and *Ros*).
- The catenoid is characterized as being the unique complete, embedded minimal surface with finite topology and two ends (*Schoen*).
Key Properties:

- Proved to be minimal by **Meusnier** in 1776.
- The helicoid has genus zero, one end and infinite total curvature.
- Together with the plane, the helicoid is the only ruled minimal surface (**Catalan**).
- It is the unique simply-connected, complete, embedded minimal surface (**Meeks** and **Rosenberg**, **Colding** and **Minicozzi**).
**Key Properties:**

- **Weierstrass Data:** $M = \mathbb{C}$, $g(z) = z$, $dh = z \, dz$.

- Discovered by **Enneper** in 1864, using his newly formulated analytic representation of minimal surfaces in terms of holomorphic data, equivalent to the Weierstrass representation.

- This surface is non-embedded, has genus zero, one end and total curvature $-4\pi$.

- It contains two horizontal orthogonal lines and the surface has two vertical planes of reflective symmetry.
Key Properties:

- Weierstrass Data: \( M = \mathbb{C} - \{0\}, \quad g(z) = z^2 \left( \frac{z+1}{z-1} \right), \)
  \[
  dh = i \left( \frac{z^2-1}{z^2} \right) \, dz.
  \]

- Found by Meeks, the minimal surface defined by this Weierstrass pair double covers a complete, immersed minimal surface \( M_1 \subset \mathbb{R}^3 \) which is topologically a Möbius strip.

- This is the unique complete, minimally immersed surface in \( \mathbb{R}^3 \) of finite total curvature \(-6\pi\) (Meeks).
Bent helicoids.

Key Properties:

- Weierstrass Data: $M = \mathbb{C} - \{0\}$, $g(z) = -z \frac{z^n + i}{iz^n + i}$, $dh = \frac{z^n + z^{-n}}{2z} dz$.
- Discovered in 2004 by Meeks and Weber and independently by Mira.
Key Properties:

- Weierstrass Data: Based on the square torus $M = \mathbb{C}/\mathbb{Z}^2 - \{(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2})\}$, $g(z) = P(z)$.
- Discovered in 1982 by Costa.
- This is a thrice punctured torus with total curvature $-12\pi$, two catenoidal ends and one planar middle end. Hoffman and Meeks proved its global embeddedness.
- The Costa surface contains two horizontal straight lines $l_1, l_2$ that intersect orthogonally, and has vertical planes of symmetry bisecting the right angles made by $l_1, l_2$. 

Key Properties:

- Weierstrass Data: Defined in terms of cyclic covers of $S^2$.

- These examples $M_k$ generalize the Costa torus, and are complete, embedded, genus $k$ minimal surfaces with two catenoidal ends and one planar middle end. Both existence and embeddedness were given by Hoffman and Meeks in 1990.
Key Properties:

- The Costa surface is defined on a square torus $M_{1,1}$, and admits a deformation (found by Hoffman and Meeks, unpublished) where the planar end becomes catenoidal.
**Genus-one helicoid.**

**Key Properties:**

- The unique end of $M$ is asymptotic to a helicoid, so that one of the two lines contained in the surface is an *axis* (like in the genuine helicoid).

- Discovered in 1993 by Hoffman, Karcher and Wei.

Singly-periodic Scherk surfaces.

Key Properties:

- **Weierstrass Data:** \( M = (\mathbb{C} \cup \{\infty\}) - \{\pm e^{\pm i \theta/2}\}, \quad g(z) = z, \quad dh = \frac{iz \, dz}{\prod (z \pm e^{\pm i \theta/2})}, \) for fixed \( \theta \in (0, \pi/2] \).

- Discovered by **Scherk** in 1835, these surfaces denoted by \( S_\theta \) form a 1-parameter family of complete, embedded, genus zero minimal surfaces in a quotient of \( \mathbb{R}^3 \) by a translation, and have four annular ends.

- Viewed in \( \mathbb{R}^3 \), each surface \( S_\theta \) is invariant under reflection in the \((x_1, x_3)\) and \((x_2, x_3)\)-planes and in horizontal planes at integer heights, and can be thought of geometrically as a desingularization of two vertical planes forming an angle of \( \theta \).
Doubly-periodic Scherk surfaces.

Key Properties:

- Weierstrass Data: $\mathcal{M} = (\mathbb{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$,
  
  $$dh = \frac{z \, dz}{\prod (z \mp e^{\pm i\theta/2})}, \quad \text{where } \theta \in (0, \pi/2] \text{ (the case } \theta = \frac{\pi}{2}).$$

- It has implicit equation $e^z \cos y = \cos x$.

- Discovered by Scherk in 1835, are the conjugate surfaces to the singly-periodic Scherk surfaces.
Key Properties:

- **Weierstrass Data**: \( M = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z^8 - 14z^4 + 1\} \),
  \[ g(z, w) = z, \quad dh = \frac{z \, dz}{w}. \]

- Discovered by **Schwarz** in the 1880’s, it is also called the P-surface.

- This surface has a rank three symmetry group and is invariant by translations in \( \mathbb{Z}^3 \).

- Such a structure, common to any triply-periodic minimal surface (TPMS), is also known as a **crystallographic cell** or **space tiling**.

Embedded TPMS divide \( \mathbb{R}^3 \) into two connected components (called **labyrinths** in crystallography), sharing \( M \) as boundary (or **interface**) and interweaving each other.
Discovered by Schwarz, it is the conjugate surface to the P-surface, and is another famous example of an embedded TPMS.
In the 1960’s, **Schoen** made a surprising discovery: another minimal surface locally isometric to the Primitive and Diamond surface is an embedded **TPMS**, and named this surface the **Gyroid**.
1860  Riemann’s discovery!

I am foliated by circles
**Riemann minimal examples.**

Image by Matthias Weber

**Key Properties:**

- Discovered in 1860 by **Riemann**, these examples are invariant under reflection in the \((x_1, x_3)\)-plane and by a translation \(T_\lambda\).

- After appropriate scalings, they converge to catenoids as \(t \to 0\) or to helicoids as \(t \to \infty\).

- The Riemann minimal examples have the amazing property that every horizontal plane intersects the surface in a circle or in a line.

- **Meeks, Pérez** and **Ros** proved these surfaces are the only properly embedded minimal surfaces in \(\mathbb{R}^3\) of genus 0 and infinite topology.
Problem: Classify all PEMS in $\mathbb{R}^3$ with genus zero.

$L\acute{\text{o}}pez-Ros, 1991$: Finite total curvature $\Rightarrow$ plane, catenoid
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$k = \#\{\text{ends}\}$

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Collin, 1997: Finite topology and $k > 1 \Rightarrow$ finite total curvature.
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COLDING-MINICOZZI, 2004: limits of simply connected minimal surfaces $= \text{minimal laminations}$.

MEEEKS-ROSENBERG, 2005: $k = 1 \Rightarrow$ plane, helicoid.

Theorem (Meeks, Pérez, Ros, 2007)

$k = \infty \Rightarrow \text{Riemann minimal examples.}$
The family $\mathcal{R}_t$ of Riemann minimal examples

Riemann's Infinite Staircase

Catenoid Soap Film

Perturbed Soap Film

Shifted wire
Cylindrical parametrization of a Riemann minimal example

Infinite cylinder
1860 Riemann’s discovery!

I am foliated by circles

Image by Matthias Weber
Cylindrical parametrization of a Riemann minimal example
Conformal compactification of a Riemann minimal example

Top End = North Pole

Bottom End = South Pole
The moduli space of genus-zero examples

Catenoid

Helicoid

Riemann

MODULI SPACE

\[ R_t = \text{Riemann Examples} \]

CATENOID

HELICOID
Riemann minimal examples near helicoid limits
A Riemann minimal example

Image by Matthias Weber
1860  Riemann’s discovery!

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