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MAXIMUM PRINCIPLES AT INFINITY

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Abstract

We prove a general maximum principle at infinity for properly immersed minimal surfaces with boundary in $\mathbb{R}^3$. An important corollary of this maximum principle at infinity is the existence of a fixed sized regular neighborhood for any properly embedded minimal surface of bounded curvature.

1. Introduction

Maximum principles play a fundamental and unifying role in the development of many deep results in geometry and analysis. Special cases of these general maximum principles, applied at infinity, have been used in an essential way in classifying genus zero minimal surfaces in $\mathbb{R}^3$ and in $\mathbb{R}^3/\Lambda$, where Λ is a discrete rank 2 subgroup (see [**12**, **15**, **17**]). The main goal of this paper is to develop general maximum principles at infinity for embedded complete minimal and constant mean curvature surfaces that include all previous results of this type and that are sufficiently powerful to have important applications to the global theory of these surfaces. Results of this type and the barrier type arguments used here first appeared in the proof of the strong halfspace theorem in [**9**] and in the isometric classification of properly embedded minimal surfaces in $\mathbb{R}^3$ (see [**4**]). A maximum principle at infinity for minimal surfaces of finite total curvature first appeared in [**11**]. A significant generalization was done in [**16**]. The explicit statement of the Maximum Principle at Infinity appears in Theorem 5.1.

Some clever barrier arguments are also used to prove maximum principles at infinity to study problems in other areas of differential geometry. Some earlier fundamental works on this topic can be found in the papers of Ni and Tam [**19**], Omari [**20**], and Yau [**25**]

Our paper is organized as follows. In Section 2 we prove that an injective minimal immersion of a complete surface of bounded Gaussian curvature into $\mathbb{R}^3$ is a proper mapping. In Section 3, we prove some

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results about the conformal structure of complete Riemannian surfaces of nonnegative Gaussian curvature. We then apply these results to give a new proof of Xavier's theorem [**24**] that the convex hull of a complete, nonflat minimal surface of bounded curvature is all of $\mathbb{R}^3$. In Section 4, we prove the maximum principle at infinity for minimal graphs over the zero section of the normal bundle of a parabolic minimal surface. The results in Sections 3 and 4 are applied in Section 5 to prove the maximum principle at infinity for stable minimal graphs. Finally, in Section 5 we reduce the proof of the general maximum principle at infinity to the case of stable minimal graphs.

Our new maximum principle at infinity for proper minimal surfaces is the strongest result of this type possible. In Section 5, we apply this maximum principle to prove that a connected, properly embedded minimal surface whose Gaussian curvature is bounded from below by a constant $-C$ has an embedded open regular neighborhood of radius $\frac{1}{\sqrt{C}}$. An immediate consequence of this Regular Neighborhood Theorem is that such a surface has at most cubical area growth with respect to the radial coordinate. A consequence of our cubical area growth theorem is that the space of connected, complete, embedded minimal surfaces with a given uniform bound on the Gaussian curvature is essentially compact in the topology of C^1-convergence in compact subsets of $\mathbb{R}^3$; this result is given in the statement of Corollary 5.5.

In Section 6, we prove that an injective immersion of a surface of constant mean curvature and bounded Gaussian curvature is a proper mapping. Also, in Section 6 we prove a type of maximum principle at infinity for constant mean curvature surfaces of bounded Gaussian curvature and apply it to show the existence of an embedded ε-tubular neighborhood on the mean convex side of the surface. As in the zero mean curvature case, the existence of the ε-tubular neighborhood implies the surface has at most cubical area growth in terms of the distance R from the origin $\mathbb{R}^3$. In Section 7, we prove that a properly immersed minimal surface of finite topology and one end that intersects a plane transversely in a finite number of components is recurrent for Brownian motion, a condition that implies the surface is parabolic (bounded harmonic functions are constant). Results in [**5**] and [**17**] imply that every complete, embedded minimal surface of finite genus and one end is properly embedded and intersects some plane transversely in a single component, and so, is recurrent for Brownian motion; in this case, our theorem in this section gives an independent proof that the surface is recurrent for Brownian motion without the additional assumption that it is embedded (see [**17**]).

Some of the results in our paper related to minimal surfaces have been found independently by Marc Soret [**23**], using somewhat different techniques. We would like to thank Brian White for some suggestions

on reducing the maximum principle at infinity for minimal surfaces to the special case of stable minimal graphs.

2. A complete, embedded minimal surface of bounded curvature is proper

In this section we will prove that a complete, embedded minimal surface of bounded Gaussian curvature is proper.

Theorem 2.1. *If $f\colon M \to \mathbb{R}^3$ is an injective minimal immersion of a complete, connected Riemannian surface of bounded Gaussian curvature, then the map f is proper.*

Proof. Since there is a lower bound on the curvature of M and $f : M \to \mathbb{R}^3$ is minimal, the principle curvatures of $f(M)$ are bounded in absolute value. Let $B_\varepsilon(p)$ denote the closed ball of radius ε centered at p. Since the second fundamental form of $f(M)$ is bounded, there exists an $\varepsilon > 0$ such that for any point $p \in \mathbb{R}^3$, every component of $f^{-1}(B_\varepsilon(p))$ is compact with a uniform bound on the area. Furthermore, for ε sufficiently small, we may also assume that every such compact component that intersects $B_{\varepsilon/2}(p)$, when pushed forward by f, is a disk and a graph over a domain in the tangent plane of any point on it. It follows that if p is a limit point of $f(M)$ coming from distinct components of $f^{-1}(B_\varepsilon(p))$, then there is a minimal disk $D(p)$ passing through p that is a graph over its tangent plane at p, and $D(p)$ is a limit of components in $f^{-1}(B_\varepsilon(p))$. Embeddedness and the usual maximum principle imply that any other such limit disk $D'(p)$ agrees with $D(p)$ near p. These observations easily imply that the closure $L(f(M))$ of $f(M)$ has the structure of a minimal lamination, i.e., it is foliated by the limits of $f(M)$ and these limits are minimal surfaces.

The immersion f is proper if and only if $L(f(M))$ has no limit leaves. Recall that a leaf L of $L(f(M))$ is a limit leaf if for some point $p \in L$ and all small $\delta > 0$, $B_\delta(p) \cap L$ contains an infinite number of disk components. (Note that it may be the case that a leaf in $L(f(M))$ is a limit leaf of itself.) Suppose now that $L(f(M))$ has a limit leaf L and we shall derive a contradiction. Let $\widetilde{L}$ denote the universal cover of L. Since the second fundamental form of $\widetilde{L}$ is also bounded, there exists a $\delta > 0$ such that the δ-neighborhood of $\widetilde{L}$, considered to be the zero section of its normal bundle, submerses under the exponential map applied to those normal vectors to $\widetilde{L}$. Thus, we can lift or pull back the lamination $L(f(M))$ to the δ-neighborhood of $\widetilde{L}$. Note we are working with the pulled back flat metric. Also note that $\widetilde{L}$, considered to be the zero section, is a limit leaf of the pulled back lamination.

We claim that $\widetilde{L}$ is stable. The proof of this property is more or less known in this setting but we outline the proof for the sake of completeness. If $\widetilde{L}$ were unstable, then there would exist a smooth, compact disk

$D \subset \widetilde{L}$ such that D is unstable. Since D is simply connected, a sequence of the sheets of the lifted lamination $\widetilde{L}(f(M))$ that limit to D can be chosen to be graphs over D. Let N be the unit normal vector field to L chosen so that the leaves of $\widetilde{L}(f(M))$ limit to D on the side where N points. Choose a positive eigenfunction f for the stability operator on D corresponding to the smallest eigenvalue r_1, which is negative. For t small and fixed, consider the normal graph $E(t) = tfN$ over D with zero boundary values. For t sufficiently small, the mean curvature vector of $E(t)$ points away from D. One sees this as follows. If L denotes the linearized operator of the mean curvature equation, then $L(f)+r_1 f = 0$ on D. So at a point x where $f(x) > 0$, we have $L(f) > 0$. Now L is the first variation of the mean curvature of the variation of D given by $x \mapsto x + tf(x)N(x)$; i.e., $L(f)(x) = \dot{H}_0(x)$. Hence, $\dot{H}_0(x) > 0$ and $H_t(x) > 0$ for $t > 0$, t small. Clearly, any sheet of $\widetilde{L}(f(M))$ that is a graph over D and is sufficiently close to D will intersect the family $E(t)$ a first time. The existence of such a first point of contact contradicts the usual comparison principle for mean curvature. This contradiction implies $\widetilde{L}$ is stable. By [**2**], [**21**] or [**8**], $\widetilde{L}$ is totally geodesic and hence L is a plane.

Since a plane in $\mathbb{R}^3$ is proper, L is not equal to $f(M)$. It follows that the plane L is disjoint from $f(M)$ and therefore $f(M)$ is contained in a halfspace of $\mathbb{R}^3$. But a theorem of Xavier [**24**] states that a complete nonflat minimal surface in $\mathbb{R}^3$ of bounded Gaussian curvature is never contained in a halfspace, which contradicts the previous statement and completes the proof of Theorem 2.1. For the sake of completeness, we now give a different proof of Xavier's theorem in the case M is embedded, which is the case of the theorem under consideration.

Suppose that M is a complete, connected, embedded minimal surface of bounded curvature contained in the halfspace $H^+ = \{x_3 > 0\}$ and suppose M is not contained in a smaller subspace. Let P_0 denote the (x_1, x_2)-plane. Since we are assuming that M is connected, it follows that M is either proper in H^+ or M has one more limit plane P_2 and M is proper in the slab between P_0 and P_2. Since M has bounded curvature, there is an $\varepsilon > 0$, such that $P_\varepsilon = \{x_3 = \varepsilon\}$ intersects M and such that the components of M in the slab defined by P_0 and P_ε submerse to P_0 under orthogonal projection to P_0. Let C denote one of these components and note that C is proper in the half open slab $S_\varepsilon = \{\varepsilon \geq x_3 > 0\}$. We claim that C is proper in $\overline{S_\varepsilon} = \{\varepsilon \geq x_3 \geq 0\}$. If not, there is a limit point $p \in P_0$ for C. Since C is proper in S_ε, it separates S_ε. However, consider an oriented vertical ray R above p and note that $R \cap C$ is a discrete set converging to p. Since C separates S_ε, the intersection signs of R with C are opposite on consecutive intersection points, where we fix an orientation on C. But this contradicts the fact

that the orientation normal of C with R has a fixed signed dot product with $(0,0,1)$. Hence, C is proper in $\overline{S}_\varepsilon$. Then the proof of the halfspace theorem in [**9**] shows C is contained in P_ε, which is impossible. This contradiction proves that M cannot be contained in H^+. This completes our proof of Xavier's theorem in the case M is embedded. We refer the reader to the end of Section 3 for a new proof of Xavier's theorem (Corollary 3.7) in the case of complete, immersed minimal surfaces of bounded curvature. q.e.d.

Remark 2.2. There exist dense, complete, triply-periodic, minimally immersed surfaces of bounded curvature in $\mathbb{R}^3$, and so injectivity is a necessary hypothesis in the statement of Theorem 2.1. In fact, consider the classical triply-periodic Schwartz P-surface $P \subset \mathbb{R}^3$. Fix $\theta \in [0, 2\pi]$. Using the symmetries of P, it is not difficult to prove [**13**] that the image surface $\widetilde{M_\theta}$ of the associate surface $M_\theta \subset \mathbb{R}^3$ of the universal cover of M is either a proper or a dense triply-periodic minimal surface in $\mathbb{R}^3$ and that except for a countable number of angles θ, $\widetilde{M_\theta}$ is dense in $\mathbb{R}^3$. (See [**14**] for the definition of associate surface.)

3. The conformal structure of complete Riemannian surfaces of nonnegative Gaussian curvature

Complete Riemannian surfaces with boundary and nonnegative Gaussian curvature need not be parabolic (see Proposition 3.3 below for several equivalent definitions of parabolic surface with boundary), even when they have zero Gaussian curvature. See the end of Section 3 of [**14**] for a sketch of the construction of such an example of a simply connected non-parabolic (hyperbolic), complete flat surface (with boundary) due to Pascal Collin. However, such flat non-parabolic surfaces come close to being parabolic as demonstrated by the following theorem, which is the main result of this section.

Theorem 3.1. *Let M be a complete Riemannian surface with non-empty boundary and with Gaussian curvature function $K\colon M \to [0,\infty)$. For $R > 0$, let $M(R) = \{p \in M \mid d(p, \partial M) < R\}$. Suppose that for each R, the restricted function $K|_{M(R)}$ is bounded. Then, for every $\delta > 0$, $\Sigma = M - M(\delta)$ is a parabolic Riemann surface in the sense that the boundary of Σ has full harmonic measure.*

In what follows we will assume that M is simply connected. This assumption does not change the validity of the theorem because a Riemannian surface with boundary has full harmonic measure if and only if its universal cover does. Note that this assumption and the fact that $K|_{M(R)}$ is bounded for every R guarantees that there exists an $\varepsilon_0 > 0$ depending on R such that the injectivity radius function I_M of M restricted to $M(R)$ satisfies $I_M(p) \geq \min\{d_M(p, \partial M), \varepsilon_0\}$. Henceforth, we

will assume $\delta < \varepsilon_0$ for some fixed R. Making this assumption is alright, since if $0 < r_1 \leq r_2$ and $M - M(r_1)$ is parabolic, then $M - M(r_2)$ is also parabolic.

The above theorem motivates the next definition.

Definition 3.2. A Riemannian surface M is δ-parabolic if for every $\delta > 0$, $\Sigma = M - M(\delta)$ is a parabolic Riemannian manifold.

Before proving the above theorem, we recall some equivalent properties for a Riemann surface to be parabolic. We summarize these properties in the next proposition.

Proposition 3.3. *Suppose M is a Riemannian surface with nonempty boundary and fix a point p in the interior of M. Let μ_p denote the associated harmonic or hitting measure associated to p. Then the following properties are equivalent.*

1. $\displaystyle\int_{\partial M} \mu_p = 1$.
2. *If $h : M \to \mathbb{R}$ is a bounded harmonic function, then $h(p) = \displaystyle\int_{\partial M} h(x)\mu_p$.*
3. *A bounded harmonic function on M is determined by its boundary values.*
4. *Let $\widetilde{M}$ denote the universal cover of M with the pulled back Riemannian metric. Then $\widetilde{M}$ is conformally equivalent to the closed unit disk with a closed set of Lebesque measure 0 removed from the boundary. In particular, there exists a proper harmonic function $h\colon \widetilde{M} \to [0, \infty)$;*

If M satisfies any of the above four properties, then M is called ***parabolic****.*

Proof. First recall the definition of the harmonic or hitting measure in the special case that M is compact. Given an interval $I \subset \partial M$, the μ_p measure of I is $\mu_p(I) =$ probability of a Brownian path in M beginning at p of arriving for a first time on ∂M at a point of I. Note that μ_p is also the harmonic measure on ∂M. In the case M is compact, almost all Brownian paths beginning at p must eventually arrive at ∂M, so property 1 always holds. Also in this case, it is straightforward to show that properties 2 and 3 also hold.

Suppose now that M is noncompact and $p \in M(1) \subset M(2) \subset \ldots$ is a compact exhaustion which induces a compact exhaustion of ∂M. The hitting measure μ_p on ∂M is the limit of the hitting measures $\mu_p(i)$ of $\partial M(i)$ restricted to ∂M. We now prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

Suppose now that 1 holds. Since $M(i)$ is compact, for a bounded harmonic function $h\colon M \to \mathbb{R}$, $h(p) = \displaystyle\int_{\partial M(i)} h(x)\mu_p(i)$. Since $\mu_p(i) \to \mu_p$ on ∂M, 2 holds. Statement 2 clearly implies statement 3.

We check that if $\int_{\partial M} \mu_p \neq 1$, then 3 fails to hold. Suppose $\int_{\partial M} \mu_p = 1-\varepsilon$ for some $\varepsilon > 0$. Define the bounded harmonic function $H\colon M \to \mathbb{R}$ by $H(q) = \int_{\partial M} \mu_q$. This bounded harmonic function has the same boundary values as the constant function 1 but $H(p) = 1-\varepsilon$ and so H is not constant. Therefore *3* fails and this contradiction proves that 3 $\Rightarrow$ 1 and so, properties 1, 2, 3 are equivalent.

Since Brownian motion of the universal covering $\pi\colon \widetilde{M} \to M$ projects to Brownian motion on M, then $\int_{\partial M} \mu_p = 1$ if and only if $\int_{\partial \widetilde{M}} \mu_{\widetilde{p}} = 1$, where $\widetilde{p} \in \pi^{-1}(p)$. The uniformization theorem implies that $\widetilde{M}$ is conformally the closed unit disk D with a closed set C removed from ∂D. Taking $\widetilde{p}$ to be the origin in D and noting that the Lebesque measure of the closed set C in ∂D is equal to 2π times $\int_C \mu_{\widetilde{p}}$, we conclude that C has Lebesque measure zero if and only if statement 1 holds.

To complete the proof of Proposition 3.3, it remains only to prove that there exists a proper harmonic function $D - C \to [0,\infty)$. Since C is compact with Lebesque measure zero, there exists collections I_n, each collection consisting of a finite number of disjoint closed intervals in ∂D which cover C and such that the total Lebesque measure of I_n is less than $\frac{1}{2^n}$.

After choosing a subsequence, we may assume that the intervals I_n are contained in the interior of the intervals of I_{n-1}. Let $h_n\colon D-C \to [0,n]$ be a smooth nonnegative harmonic function which is zero on $\partial D - \cup I_{n-1}$ and n on $\cup I_n$.

Note that $H(k) = \sum_{i=2}^{k} h_i$ is a sequence of smooth harmonic functions which are uniformly bounded at the origin and which converge to a smooth, proper harmonic function $h\colon D-C \to [0,\infty)$, which completes the proof of the proposition. q.e.d.

Proof. We now return to the proof of Theorem 3.1. Consider the function $d_{\partial M}\colon M \to [0,\infty)$ defined by $d_{\partial M}(p) =$ distance of p to ∂M. We will show that $\ln(d_{\partial M})\colon (M - \partial M) \to \mathbb{R}$ is superharmonic by proving that $\ln(d_{\partial M})$ satisfies the mean value inequality for superharmonic functions.

Let $p \in M$ and let γ be a minimizing geodesic joining p to $y \in \partial M$. Suppose for the moment that p is not a conjugate point to y along γ. In this case, there exist a small disk neighborhood D of p such that for every point in D there is a unique geodesic joining this point to y and close to γ. Let $L(x)$ be the length of the geodesic joining y to x and note that L is a smooth function on D.

Since M has nonnegative Gaussian curvature, the Laplacian comparison theorem implies $\Delta L(x) \leq \frac{1}{L(x)}$. It follows that $\Delta \ln(L)(x) \leq 0$ for $x \in D$. Also, $\ln(L)(p) = \ln(d_{\partial M})(p)$ and $d_{\partial M} \leq L$ on D. Hence, $\ln(d_{\partial M})$ satisfies the mean value inequality for superharmonic functions at the point p.

Suppose now that p and y are conjugate along γ. Let z be on γ and a small distance from y. Since p is not conjugate to z along γ, the previous calculation shows that there is a neighborhood $\widetilde{D}$ of p such that the distance function $\widetilde{L}$ from points in $\widetilde{D}$ to z is smooth and $\ln(\widetilde{L})$ is superharmonic in $\widetilde{D}$. Since y and p are conjugate along γ, for z sufficiently close to y, we have $\Delta\widetilde{L} < 0$. It follows that the function $\ln(\widetilde{L}+d(y,z))$ is superharmonic in some small neighborhood of p. Since $\ln(\widetilde{L}+d(y,z)) \geq \ln(d_{\partial M})$ and $\widetilde{L}+d(x,y) = d_{\partial M}$ at p, then $\ln(d_{\partial M})$ satisfies the mean value inequality for superharmonic functions at p. Hence, $\ln(d_{\partial M})$ is superharmonic on $M - \partial M$. q.e.d.

Lemma 3.4. *Let M and Σ be as in the statement of Theorem* 3.1. *Fix a point $q \in \Sigma$. For $R > \delta$ and $T > 0$, let $\Gamma(T,R)$ = set of Brownian paths in M beginning at q and which enter the neighborhood $M(R)$ for some $t \geq T$ but do not ever intersect $\partial\Sigma$. Let μ be the Wiener measure on the set of these Brownian paths. Then* $\lim_{T\to\infty} \mu(\Gamma(T,R)) = 0$.

Proof. Fix some $R > \delta$ and $T_1 > 0$. We will show that there exists a $\varepsilon > 0$ (independent of T_1) that only depends on R, such that there exists a $T_2 > T_1$ and $\mu(\Gamma(T_2,R)) \leq (1-\varepsilon)\mu(\Gamma,T_1)$. Repeating this construction n times shows that there exists at T_n with $\mu(\Gamma(T_n,R)) \leq (1-\varepsilon)\mu(\Gamma(T_{n-1},R)) \leq (1-\varepsilon)^n\mu(\Gamma,T_1)$. This will imply the lemma.

Let $\gamma_0(p)$ be the points on a minimizing geodesic joining p to $\pi(p) \in \partial M$, a distance greater than or equal to $\delta/2$ from $\pi(p)$. The injectivity radius function of $M(2R)$ is bounded away from 0, by a positive $b = \varepsilon(R)/2$ say. So $\gamma_0(p)$ can be extended beyond p a distance b to a geodesic $\gamma(p)$. We can assume $b \leq \delta/2$.

Let W be the normal bundle of $\gamma(p)$ of radius $b/2$ with the induced metric and consider $\gamma(p)$ to be the zero section of this bundle. Since the image of W under the exponential map lies in $M(2R)$, the induced metric on W is c-quasi-isometric to the rectangle $[0, \text{length}(\gamma(p))] \times [0, b/2]$. Furthermore, the constant c can be chosen independent of p in $M(R) \cap \Sigma$. It follows that the measure of the Brownian paths beginning at $p \in W$ and leaving W for a first time at the side of W closest to $\pi(p)$ at a time $t, t \leq 2R$, is at least a fixed number $\tau > 0$. This is true since W is uniformly quasi-isometric to a flat rectangle whose height is fixed and whose base is bounded. Hence, the measure of the set of Brownian paths beginning at p in M, and going to the boundary of Σ at a time $t \leq 2R$ is at least τ.

Now choose T' large enough so that at least half of the Brownian paths in $\Gamma(T_1, R)$ enter $M(R)$ between times T_1 and T'. Let $T_2 = T' + 2R$ and $\varepsilon = \frac{\tau}{2}$. Of the paths in $\Gamma(T_1, R)$ that enter $M(R)$ between the times T_1 and T', τ of them go to the boundary in time at most $2R$. Hence, τ of them do not lie in $\Gamma(T_2, R)$. Hence, $\mu(\Gamma(T_2, R) \leq (1 - \frac{\tau}{2})\mu(\Gamma(T_1, R))$. This completes the proof of the lemma. q.e.d.

Corollary 3.5. *Let M and Σ be the surfaces defined in Theorem 3.1. Fix an interior point $q \in \Sigma$ and $R > \delta$. For $n \in \mathbb{N}$, let $D(n)$ be a compact exhaustion of Σ by balls of radius n centered at q. Let $C(n) = (\partial D(n) \cap M(R)) - \partial\Sigma$ and let $\mu_q(n)$ be the corresponding harmonic measure of $\partial D(n)$. Then $\displaystyle\int_{C(n)} \mu_q(n)$ tends to zero as n goes to infinity.*

Proof. Choose $\tau > 0$. Choose k large enough so that $\displaystyle\int_{\partial D(k)\cap\partial\Sigma} \mu_q(k) < \int_{\partial\Sigma} \mu_q - \frac{\tau}{2}$. By the previous lemma, we can choose T large enough so that $\mu(\Gamma(T, R)) < \frac{\tau}{2}$. Since the Gaussian curvature of Σ is nonnegative, the subset of image Brownian paths of time less than or equal to T is contained in some $D(n)$, where $n > k$. It follows from the definition of $\Gamma(T, R)$ that $\displaystyle\int_{C(n)} \mu_q(n) < \frac{\tau}{2} + \mu(\Gamma(T, R)) < \tau$, which proves the corollary. q.e.d.

We now complete the proof of Theorem 3.1. If Σ is not parabolic, then $\displaystyle\int_{\partial\Sigma} \mu_q \neq 1$ for some interior point $q \in \Sigma$. We will derive a contradiction by constructing a sequence of positive bounded harmonic functions $h(n)$ on the geodesic balls $D(n) \subset \Sigma$ that lie below a fixed superharmonic function but for which the values of $h(n)(p)$ are unbounded. Fix $R > \delta$ and let $C(n) = (\partial D(n) \cap M(R)) - \partial\Sigma$.

Let $G\colon \Sigma \to [0, \infty)$ be the restricted superharmonic function $G = \ln(d_{\partial M}) - \ln(\delta)$, which is zero on $\partial\Sigma$. Let $\mu_q(n)$ be the harmonic measure on $\partial D(n)$. Suppose $\int_{\partial\Sigma} \mu_q = 1 - a$ for some $a > 0$.

By the Corollary 3.5, there exists positive integer N depending on R, so that for $n \geq N$, then $\displaystyle\int_{C(n)} \mu_q(n) < \frac{a}{2}$. Since

$$\int_{\partial D(n)-\partial\Sigma} \mu_q(n) > 1 - \int_{\partial\Sigma} \mu_q = a,$$

then, for $n \geq N$, $\displaystyle\int_{\partial D(n)-M(R)} \mu_q(n) > \frac{a}{2}$. Let $h(n, R)$ be the harmonic function on $D(n)$ which has boundary values $\ln(R) - \ln(\delta)$ on $\partial D(n) -$

$M(R)$ and zero elsewhere. Then for $n \geq N$:

$$h(n,R)(q) = (\ln(R) - \ln(\delta)) \int_{\partial D(n) - M(R)} \mu_q(n) \geq \frac{a}{2}(\ln(R) - \ln(\delta)).$$

Since $h(n,R)$ has boundary values below the superharmonic function G, then $h(n,R)(q) \leq G(q)$. This is impossible for large n and R. This contradiction completes the proof of the theorem. q.e.d.

Corollary 3.6. *Suppose M is a complete, orientable stable minimal surface with boundary in $\mathbb{R}^3$ with a positive Jacobi function J. If $J \geq \varepsilon$, for some $\varepsilon > 0$, then M is δ-parabolic.*

Proof. First note that a Riemannian surface W is δ-parabolic if and only if for all $\delta' > 0$, the surface $W - W(\delta')$ is also δ-parabolic. Thus, without loss of generality, we may assume that M has the form $W - W(\delta')$ for some $\delta' > 0$ and where W is a stable minimal surface with a positive Jacobi function $J_W \geq \varepsilon$. In particular, by Schoen's curvature estimate for stable, orientable minimal surfaces, we may assume that M has bounded Gaussian curvature. After multiplying J by $\frac{1}{\varepsilon}$, we may assume that $J \geq 1$. Consider the new Riemannian manifold $\widetilde{M}$, which is the manifold M with the new metric $\widetilde{\langle \cdot, \cdot \rangle} = J\langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on M. Since $J \geq 1$ and parabolicity is a conformal property in dimension two, M is δ-parabolic if and only if $\widetilde{M}$ is δ-parabolic; we will prove that $\widetilde{M}$ is δ-parabolic.

If $K, \widetilde{K}$ denote the Gaussian curvature functions for $M, \widetilde{M}$, respectively, then the Jacobi equation gives:

$$\widetilde{K} = \frac{K - \frac{1}{2}\Delta_M(\ln J)}{J} = \frac{1}{2}\frac{|\nabla J|^2}{J^3}$$

Choose $\delta > 0$ and let $\widetilde{\Sigma} = \widetilde{M} - \widetilde{M}(\delta)$. Let $\Sigma \subset M$ be the submanifold corresponding to $\widetilde{\Sigma}$. By the Harnack inequality [**10**], $\frac{|\nabla J|}{J}$ is bounded, and so one has that $\widetilde{K}$ is nonnegative and bounded on $\widetilde{\Sigma}$. It then follows from Theorem 3.1 that $\widetilde{\Sigma}$ is parabolic, which means that Σ is parabolic. Since $M - M(\delta) \subset \Sigma$ and Σ is parabolic, $M - M(\delta)$ is parabolic. This completes the proof that M is δ-parabolic. q.e.d.

Corollary 3.7 (Xavier's Theorem)**.** *The convex hull of a complete, connected minimal surface M of bounded Gaussian curvature in $\mathbb{R}^3$ is either a plane or all of $\mathbb{R}^3$.*

Proof. Suppose the theorem fails and that M is not a plane. In this case the convex hull of M is not a plane as well and so, there exists a plane P_0 that is disjoint from M. Without loss of generality, we may assume that P_0 is the (x_1, x_2)-plane and that $H^+ = \{x_3 > 0\}$ is a smallest halfspace containing M.

Since M has bounded curvature, there is an $\varepsilon > 0$ such that for every component C of M in the slab between P_0 and $P_\varepsilon = \{x_3 = \varepsilon\}$, the linear Jacobi function $\langle e_3, N_C\rangle \geq \frac{1}{2}$, where N_C is the unit normal to C, after making a choice of orientation of C. Note that C satisfies the hypotheses of the surface described in Corollary 3.6. Choose a positive $\delta < \varepsilon$ so that $C(\delta) = \{p \in C \mid x_3(p) \leq \delta\}$ is nonempty and note that $C(\delta)$ is parabolic by Corollary 3.6. But $x_3 \colon C(\delta) \to (0, \delta]$ is a bounded harmonic function with the same boundary values as the constant function δ. Hence x_3 is constant on $C(\delta)$, which is a contradiction because $C(\delta)$ is not contained in a plane. This completes our proof of the corollary. q.e.d.

4. Maximum Principle at Infinity for parabolic minimal graphs

In this section we prove the maximum principle at infinity for parabolic minimal graphs of bounded curvature. This was proved by M. Soret in [**22**].

Let M_1 be an embedded parabolic minimal surface and M_2 a small graph over M_1 in the normal bundle of M_1 (here we consider M_1 to be the zero section of some small normal interval bundle which has a flat induced metric under the exponential map). We denote by u the graph function, $u > 0$, and we assume M_1, M_2 have bounded curvature. We wish to prove that if $u \geq c > 0$ on ∂M_1, then $u \geq \widetilde{c}$ on M_1, for some $\widetilde{c} > 0$.

Since M_1 and M_2 are disjoint and of bounded curvature, the gradient ∇ of u and the Hessian $Hess(u)$ tend to zero as $u \to 0$. More precisely, for each $\delta < c$, there exists a constant B (depending on the curvature bounds) such that if $u(p) < \delta$, then $\sup\{|\nabla u(p)|, |Hess(u(p))|\} \leq B\delta$. Now if u tends to zero, then for $\delta < c$, a part of M_2 is a graph over $M_1(\delta) = \{p \in M_1 \mid u(p) \leq \delta\}$. We can assume $M_1(\delta)$ is connected by working with one of the components. On $M_1(\delta), u = \delta$ on $\partial M_1(\delta)$ and $u \leq \delta$ in the interior.

Consider the metric $\overline{ds}$ on $M_1(\delta)$ which is the pull back by u of the metric on M_2. Since ∇u is bounded, this metric is quasi-isometric to the metric ds of M_1, so $M_1(\delta)$ is also parabolic in the $\overline{ds}$ metric. By Lemma 5.1 of [**22**], we know that there exists a $d > 0$ such that on $M_1(\delta), \overline{\Delta} u \leq d|\overline{\nabla} u|^2$. (We remark that $u < 0$ in [**22**]). Since M_1 is parabolic, $M_1(\delta)$ is too, and so there exists a proper positive harmonic function ϕ on $M_1(\delta)$, $\phi \geq 1$ on $\partial M_1(\delta)$. (The existence of such a proper harmonic function was proved in statement 4 of Proposition 3.3 in the case $M_1(\delta)$ is simply connected, which, by lifting all the data to the universal cover of $M_1(\delta)$ suffices in the following argument.) Let $\psi = \ln \phi$, then $\overline{\Delta}\psi = -\frac{|\overline{\nabla}\phi|^2}{\phi^2} = -|\overline{\nabla}\psi|^2$.

Consider the function

$$h = u + \varepsilon\psi,$$

where ε is chosen less than $\min\{1, \frac{1}{d}\}$. We now check that h achieves its minimum value on $\partial M_1(\delta)$. The function ψ is a proper positive function and u is bounded so there is a minimum p of h on $M_1(\delta)$. If this were an interior minimum, then $\overline{\nabla} u = -\varepsilon\overline{\nabla}\psi$ at p and $0 \leq \overline{\Delta} h(p) = \overline{\Delta} u - \varepsilon|\overline{\nabla}\psi|^2 \leq d|\overline{\nabla} u|^2 - \varepsilon|\overline{\nabla}\psi|^2 \leq d|\overline{\nabla} u|^2 - \frac{1}{\varepsilon}|\overline{\nabla} u|^2 = (d - \frac{1}{\varepsilon})|\overline{\nabla} u|^2 < 0$.

So p must be on $\partial M_1(\delta)$. Since there is some $q \in M_1(\delta)$ where $u(q) < \delta$ and the above inequality holds for all small $\varepsilon > 0$, then $h = u + \varepsilon\psi$ will not have its minimum on $\partial M_1(\delta)$ for ε sufficiently small, which completes the proof of the Maximum Principle at Infinity for parabolic minimal graphs of bounded curvature.

5. The Maximum Principle at Infinity for proper minimal surfaces

In this section we prove the general maximum principle at infinity holds for properly immersed minimal surfaces with boundary. We also apply this maximum principle to prove the existence of a fixed size tubular neighborhood for any properly embedded minimal surface in $\mathbb{R}^3$ with bounded Gaussian curvature. We next apply this regular neighborhood theorem to obtain further applications.

Theorem 5.1 (Maximum Principle at Infinity)**.** *Suppose $f_1 \colon M_1 \to \mathbb{R}^3$, $f_2 \colon M_2 \to \mathbb{R}^3$ are disjoint, proper minimal immersions of surfaces with boundary, and at least one of these surfaces has nonempty boundary. Then the distance between M_1 and M_2 is equal to*

$$d(M_1, M_2) = \min\{d(M_1, \partial M_2), d(M_2, \partial M_1)\}.$$

Proof. We shall first show that if Theorem 5.1 fails to hold, then it fails to hold for two surfaces M_1', M_2', where M_1', M_2' have bounded curvature and are stable. Actually, the new M_1', M_2' are contained in some flat three-manifold N^3 that isometrically submerses to $\mathbb{R}^3$, $d(M_1', M_2') = 0$, and $d(\partial M_1' \cup \partial M_2', \partial N^3)$ is positive; we shall then obtain a contradiction in this more general setting. We will obtain these two surfaces by solving Plateau problems in region between M_1 and M_2.

Arguing by contradiction, assume that the theorem fails for some M_1, M_2. Without loss of generality, we may assume that the boundary of each of these surfaces is nonempty. After a translation of M_1, we may assume that $d(M_1, M_2) = 0$ and $\min\{d(M_1, \partial M_2), d(M_2, \partial M_1)\} > 0$. The first step in the proof is to demonstrate that M_1 and M_2 can be "separated" by a hypersurface F that is minimal and stable away from a small neighborhood of $\partial M_1 \cup \partial M_2$. The stability of F implies, by curvature estimates, that this stable, minimal surface portion of F has bounded curvature, a property which is used to construct a further

barrier and the new M_1', M_2' referred to in the previous paragraph. We now construct F.

Without loss of generality, we may assume that the boundaries of M_1 and M_2 are smooth by coming slightly in from their boundaries. Choose an $\varepsilon_0 > 0$ with $10\varepsilon_0 < \min\{d(M_1, \partial M_2), d(M_2, \partial M_1)\}$ and such that the ε_0-neighborhood $B_{\varepsilon_0}(\partial M_i)$ is a piecewise smooth manifold with boundary for $i = 1, 2$. Let $W_i(\varepsilon_0) = M_i \cup B_{\varepsilon_0}(\partial M_i)$ for $i = 1, 2$.

Let $\mathcal{F}$ be the collection of proper hypersurfaces in $\overline{\mathbb{R}^3 - W_1(\varepsilon_0) \cup W_2(\varepsilon_0)}$ which are disjoint from $M_1 \cup M_2$ and separate M_1 and M_2. Let $\widetilde{F} \in \mathcal{F}$ be a surface in $\mathcal{F}$ of least-area in the sense that compact subdomains minimize area in the restricted region. Assume, by an appropriate small choice of ε_0, that $W_1(2\varepsilon_0) \cup W_2(2\varepsilon_0)$ is a piecewise smooth manifold and $F = \widetilde{F} \cap \overline{\mathbb{R}^3 - W_1(2\varepsilon_0) \cup W_2(2\varepsilon_0)}$ is a properly embedded, least-area minimal surface with boundary in $\partial(W_1(2\varepsilon_0) \cup W_2(2\varepsilon_0))$ and $F \cap (M_1 \cup M_2) = \emptyset$. Note that F is orientable and has uniformly bounded curvature by Schoen's curvature estimates for stable minimal surfaces.

Choose an ε, $0 < \varepsilon < \varepsilon_0$, sufficiently small so that $\exp\colon N_\varepsilon(F) \to \mathbb{R}^3$ is a submersion where $N_\varepsilon(F)$ denotes the closed ε-disk bundle of the normal bundle of F. Note that $(\partial M_1 \cup \partial M_2) \cap \exp(N_\varepsilon(F)) = \emptyset$, because $d(\partial M_1 \cup \partial M_2, F) \geq \varepsilon_0$ and $\varepsilon < \varepsilon_0$. It follows that $\partial(\exp^{-1}(M_1 \cup M_2)) \subset \partial(N_\varepsilon(F))$. From now on we will consider $N_\varepsilon(F)$ to be a flat three-manifold with respect to the pulled back Riemannian metric and we will consider F to be the zero section of this bundle. F separates $N_\varepsilon(F)$ into two regions, $N_\varepsilon^+(F)$ and $N_\varepsilon^-(F)$, which respectively we refer to as the regions above and below F. For $i = 1, 2$, let $M_i^+ = \exp^{-1}(M_i) \cap N_\varepsilon^+(F)$ and $M_i^- = \exp^{-1}(M_i) \cap N_\varepsilon^-(F)$. Let γ be a line segment of length less than ε that joins a point of M_1 to a point of M_2. Since γ joins a point of M_1 to a point of M_2, γ intersects F in $\mathbb{R}^3$ by the separation property defining $\widetilde{F}$. It follows that there is a component of $\exp^{-1}(M_1)$ and a component of $\exp^{-1}(M_2)$ such that these components lie on opposite sides of $F \subset N_\varepsilon(F)$ and $\exp^{-1}(\gamma)$ contains a lift of γ that joins these two components. It follows, after possibly interchanging indices, that we may assume $d(M_1^+, M_2^-) = 0$. For simplicity of notation we let $M_1 = M_1^+$ and $M_2 = M_2^-$ and keep in mind that M_1 lies above F and M_2 lies below F in $N_\varepsilon(F)$.

When projected into $\mathbb{R}^3$, each boundary component of F comes close to ∂M_1 in $\mathbb{R}^3$ or to ∂M_2 in $\mathbb{R}^3$. More precisely, for $i = 1, 2$, let ∂_i be the collection of boundary components of F such that $\exp(\partial_i) \subset \partial B_{2\varepsilon_0}(\partial M_i)$. Choose δ, $0 < \delta < \varepsilon$. Define $F(i, \delta) = \{p \in F \mid d(p, \partial_i) \leq \delta\}$ and let $NF(i, \delta)$ denote the restriction of $N_\varepsilon(F)$ to $F(i, \delta)$. Let C be a compact stable minimal catenoid bounded by two horizontal circles and symmetric with respect to the origin O. Let $\overline{C}$ denote the convex hull of C in $\mathbb{R}^3$. Since F has bounded curvature, we can choose C sufficiently small so that for any point $p \in F(i, \delta)$ with $d(p, \partial_i) = \frac{\delta}{2}$,

$\overline{C}$ isometrically embeds in $NF(1,\delta)$ so that the origin O corresponds to p and the tangent plane T_pF corresponds to the horizontal plane of symmetry of $\overline{C}$ in $\mathbb{R}^3$. Let $\overline{C}(p)$ denote the image of such an isometric embedding. Since the curvature of F is bounded, for some positive $\eta < \varepsilon$, the top and bottom bounding disks of $\overline{C}(p)$ lie in the complement of $N_\eta(F)$ in $N_\varepsilon(F)$.

Consider the solid barrier $B = \bigcup_{p\in F(1,\delta)} \overline{C}(p) \cap N_\eta(F)$, where $d(p,\partial_1) = \frac{\delta}{2}$. Again let $M_i = M_i \cap N_\eta(F)$, as we will be working now in $N_\eta(F)$. Note that by our previous choices $M_2 \cap NF(1,\delta) = \emptyset$. Since the barrier B has a minimal thickness $\sigma > 0$, if Σ is a connected surface in a component of $N_\eta(F) - (F \cup B)$ that intersects M_2 nontrivially, then $d(\Sigma, \partial M_1) > \sigma$ in $N_\eta(F)$.

Choose a component C_1 of M_1 and a component C_2 of M_2 such that $d(C_1, C_2)$ is much less than σ. Let X_2 denote the closure of the component of $N_\varepsilon^-(F) - (F \cup B \cup C_2)$ having points of both F and C_2 on its boundary; similarly, define X_1. Let C_i denote the proper subdomain in ∂X_i corresponding to points of C_i for $i = 1, 2$.

Replace C_i by a least-area surface $\widetilde{C}_i$ in X_i with $\partial\widetilde{C}_i = \partial C_i$ and $\widetilde{C}_i$ homologous (with respect to locally finite chains) to C_i in X_i. Let $\widetilde{F}$ be the closure of the component of $F - (F(1,\frac{\sigma}{2}) \cup F(2,\frac{\sigma}{2}))$. Since $N_\varepsilon(F)$ submerses in $\mathbb{R}^3$, translation is well-defined in $N_\varepsilon(F)$. Define $M_1' = \widetilde{C_1} \cap N_{\frac{\eta}{2}}(\widetilde{F})$ and $\widetilde{M_2} = \widetilde{C_2} \cap N_{\frac{\eta}{2}}(\widetilde{F})$. Consider M_1' and $\widetilde{M_2}$ to be contained in $N_\varepsilon(F)$. By our choices of ε_0, ε, ∂, η and σ, there is enough room in $N_\varepsilon(F)$ to translate $\widetilde{M_2}$ no further than $d(C_1, C_2)$ which is much smaller than σ, so, that the translated surface M_2' and the surface M_1' are disjoint, $d(M_1', M_2') = 0$, $\min\{d(M_1', \partial M_2'), d(M_2', \partial M_1')\} > \frac{\sigma}{10}$, and for $i = 1, 2$, M_i' is stable and has bounded curvature and $d(M_i, \partial N_\varepsilon(F)) > 0$.

This completes the proof of our claim at the beginning of the proof of the theorem. Thus, we will henceforth assume that M_1, M_2 are both stable, analytic and have bounded curvature. For future calculations, it is convenient to relabel these surfaces as M_0, M_1. Similar arguments as those given above, show after replacing by subdomains that we may consider M_0 to be the zero section of $N_\varepsilon(M_0)$ and M_1 to be the positive normal graph of a function $\varphi\colon M_0 \to (0,\varepsilon)$ with $\varphi|_{\partial M_0} = \varepsilon$. After a homothety of the metric, we may assume that $\varepsilon = 1$, and so, the boundary of M_1 is at height 1 over the boundary of M_0 at height 0.

It remains to prove that there is a stable minimal surface $\Sigma \subset N_1(M_0)$ which is the graph of a positive function on M_0 with constant boundary values, which is not bounded away from zero and which is δ-parabolic. In order to do this, we first observe that, without loss of generality, we may assume that the absolute geodesic curvature of ∂M_0 is bounded

(but one needs to replace M_1 by $M_1' \subset M_1$ and M_0 by $M_0' \subset M_0$ so that M_1' is a graph over M_0' and where $\varphi|_{\partial M_0'}$ is the almost constant function 1). This assumption of bounded geodesic curvature on ∂M_0 is elementary to show to hold, once one notes that for every $R > 0$, there exists a $\delta(R) > 0$ such that for all $p \in M_0$ with $d_M(p, \partial M_0) = R$, then the graphing function φ satisfies $\varphi(p) \geq \delta(R)$ (standard compactness argument). (Actually what one does to prove the bounded geodesic curvature property for ∂M_0 is to first contract M_0 a fixed small distance from its boundary to obtain $M_0' \subset M_0$. Then one adds back onto M_0' a locally well-defined smaller fixed size neighborhood in M_0 to obtain an abstractly defined analytic Riemannian surface M_0'' with $C^{1,1}$ boundary which satisfies a uniform geodesic curvature estimate almost everywhere. The surface M_1 can now be viewed as a graph M_1'' in the associated normal bundle to M_0''. After a small perturbation of $\partial M_0''$ inside M_1'', then the surface has bounded geodesic curvature.)

Note that we may assume by stability that the metrics of M_0 and of M_1 are almost-flat and that the gradient of φ is almost zero. We now assume these properties for M_0, M_1 and φ and show how to construct a new minimal graph $M(h)$ between M_0 and M_1, for some $h \in [0,1]$, which is δ-parabolic and has constant boundary values. Note that under our combined normalizations that the boundary of M_0 can be assumed to have almost zero geodesic curvature (choose ε very small and apply the uniform bound on the geodesic curvature of ∂M_0 and then expand the metric by $\frac{1}{\varepsilon}$).

We claim that there exists a collection $M' = \{M_t\}_{t\in[0,1]}$ of disjoint minimal graphs over M_0 with $\partial M_t = \partial M_0 \times \{t\} \subset N_1^+(M_0) = M_0 \times [0,1]$ (in natural normal coordinates); this collection will be a subset of a related collection $\mathcal{M}$. Using appropriate barriers, $M_{\frac{1}{2}}$ is constructed to be a surface of least-area in $N_1^+(M_0)$ between M_0 and M_1. Appropriate barriers here are the unique, almost flat minimal annuli that are close to ∂M_1 and which have their boundary being $\partial_{\frac{1}{2}}$ and the 1-parallel curves in M_0, M_1, respectively, which are parallel to $\partial M_0, \partial M_1$, respectively. It is straightforward to check that $M_{\frac{1}{2}}$ is a graph of uniformly bounded gradient over M_0. Then, one obtains a similar graph $M_{\frac{1}{4}}$ between M_0 and $M_{\frac{1}{2}}$ and a similar graph $M_{\frac{3}{4}}$ between $M_{\frac{1}{2}}$ and M_1. Barrier arguments again give a uniform bound on the gradient of the graphing functions. Continuing this process, yields a collection $\{M_t\}$ of graphs over M_0 with $t \in \{\frac{k}{2^n} \mid 0 \leq k \leq 2^n, n \in \mathbb{N}\}$. Taking the closure of this set of graphs yields a collection of minimal graphs $\mathcal{M}$ with a uniform bound on the gradient of their graphing functions and these graphs are disjoint in their interiors.

For what follows, it is important to find an appropriate indexing set for the surfaces in $\mathcal{M}$. Fix a point $p \in M_0$ of distance 1 from ∂M_0. Each

minimal graph $M \in \mathcal{M}$ is determined by $h = h(M) \in [0,1]$ equal to the height of M over the point $p \in M_0$; in this case we will denote M by $M(h)$. Let H be the set of all these heights and note that H is an ordered compact subset of $[0,1]$. Let $F\colon H \to [0,1]$ be the onto function defined by $F(h)$ is the constant boundary value of the corresponding surface $M(h) \in \mathcal{M}$. Note that F is at most 2 to 1.

Our next goal is to use the function F to construct a positive Jacobi function on one of the surfaces $M(h)$ in $\mathcal{M}$. We will then use this positive Jacobi function to prove that $M(h)$ is δ-parabolic, thereby, obtaining our desired contradiction. Since F is order preserving ($h_1 \leq h_2 \Rightarrow F(h_1) \leq F(h_2)$), an elementary argument shows that there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subset H$ converging to some fixed $h \in H$, and so that

$$|F(h_{n+1}) - F(h_n)| \geq |h_{n+1} - h_n|$$

for all $n \in \mathbb{N}$. [This elementary argument goes as follows. Let $h_1 = h(M_{\frac{1}{2}})$. If $h(M_{\frac{1}{2}})$ is closer to $h(M_0)$ than to $h(M_1)$, define $h_2 = h(M_{\frac{1}{4}})$; otherwise, let $h_2 = h(M_{\frac{3}{4}})$. Once one defines h_1, h_2, then one defines h_3 and continues defining h_n inductively. For example, suppose that $h_2 = h(M(\frac{1}{4}))$. If $h(M(\frac{1}{4}))$ is closer to $h(M(\frac{1}{2})$ than to $h(M_0)$, then define $h_3 = h(M(\frac{3}{8}))$; otherwise, $h_3 = h(M(\frac{1}{8}))$.] The sequence of paired graphing functions H_n, H_{n+1} of the surfaces $M(h_n)$ and $M(h_{n+1})$ over $M(h)$, under absolute difference and normalized by multiplying by a constant c_n so as to have the function have the value 1 at the point $p_h \in M(h)$ above the point p, yield in the limit a positive Jacobi function J on $M(h)$ with $J(p_h) = 1$. The Jacobi function J is bounded away from zero along $\partial M(h)$ because of the above inequality and the fact that the gradient of the graphing function for all of the surfaces in $\mathcal{M}$ is uniformly bounded in any fixed size neighborhood of ∂M_0.

By the Harnack inequality, $\frac{|\nabla J|}{J}$ is bounded from above in any fixed size neighborhood of $\partial M(h)$, and so, J is also bounded from above and away from zero in any fixed size neighborhood of $\partial M(h)$. Consider the metric $\widetilde{\langle\cdot,\cdot\rangle} = J\langle\cdot,\cdot\rangle$ on $M(h)$ and let $\widetilde{M}(h)$ denote this new Riemannian surface and let $\widetilde{K}$ denote its Gaussian curvature. Since $|\nabla J|$ is bounded in any fixed size neighborhood of $\partial M(h)$, the metric of $\widetilde{M}(h)$ is complete in some fixed σ-neighborhood of its boundary for some $\sigma > 0$ (incomplete geodesics in $\widetilde{M}(h)$ do not lie in σ-neighborhood of $\partial\widetilde{M}(h)$ unless they hit $\partial\widetilde{M}(h)$). By the formula given in the proof of Corollary 3.6,

$$\widetilde{K} = \frac{1}{2}\frac{|\nabla J|^2}{J^3},$$

and so it is a nonnegative function. By the Harnack inequality, $\widetilde{K}$ is bounded on $M(R)$ for any $R > 0$.

In [**7**], Fischer-Colbrie showed that the metric on $\widetilde{M}(h)$ is complete if $\partial\widetilde{M}(h)$ were compact. In fact, since M_h is complete and J is bounded away from zero on $M_h(1)$, geodesics only become incomplete at distances at least a fixed positive distance from $\partial\widetilde{M}(h)$, and so, her argument applies to our case too to show that $\widetilde{M}(h)$ is complete. In fact, using the fact that $\widetilde{K}$ is bounded on some fixed size neighborhoods of both $M(h)$ and $\widetilde{M}(h)$, Fischer-Colbrie's argument can easily be modified to show that the distance function to $\partial M(h)$ in $M(h)$ and the distance function to $\partial\widetilde{M}(h)$ in $\widetilde{M}(h))$ are comparable in the following sense: any fixed size neighborhood of the boundary of one of these surfaces is contained in a fixed size neighborhood of the boundary of the other surface. Hence, the function $\widetilde{K}$, which is bounded on $M(h)(R)$ for any R, is bounded on $\widetilde{M}(h)(R)$ for any R. Thus, Theorem 3.1 implies $\widetilde{M}(h)$ is δ-parabolic. Since J is bounded away from zero on $M(h)(1)$, it follows that $M(h)$ is also δ-parabolic.

This completes the proof of Theorem 5.1. q.e.d.

Corollary 5.2. *The Maximum Principle at Infinity holds for properly immersed minimal surfaces in any complete flat three-manifold.*

Proof. Suppose N^3 is a complete flat three-manifold and M_1, M_2 are two disjoint properly immersed minimal surfaces in W^3 where either M_1 or M_2 has nonempty boundary. Suppose $d(M_1, M_2) + \varepsilon = \min\{d(\partial M_1, M_2), d(\partial M_2, M_1)\}$ for some $\varepsilon > 0$. In this case choose a geodesic segment γ joining a point of M_1 to a point of M_2 of length less than $d(M_1, M_2) + \frac{\varepsilon}{2}$. Let $\pi\colon \mathbb{R}^3 \to N^3$ denote the universal cover of N^3 and let L be a component of $\pi^{-1}(M_1 \cup M_2 \cup \gamma)$. Let $\widetilde{M_1}$ be a component of $L - \pi^{-1}(\mathrm{Int}(\gamma))$ that covers M_1. Let $\widetilde{\gamma}$ be a lift of γ to $\mathbb{R}^3$ such that an end point of $\widetilde{\gamma}$ lies on M_1. Let $\widetilde{M_2}$ be a component of $L - \pi^{-1}(\mathrm{Int}(\gamma))$ such that $\widetilde{M_2}$ covers M_2 and the other end point of $\widetilde{\gamma}$ lies on $\widetilde{M_2}$.

Since the maximum principle at infinity holds in $\mathbb{R}^3$ (Theorem 5.1) and $d(\widetilde{M_1}, \widetilde{M_2}) \leq \mathrm{length}(\widetilde{\gamma}) < \frac{\varepsilon}{2} + d(M_1, M_2)$, then without loss of generality we can assume there exists a line segment α joining a point of $\partial\widetilde{M_1}$ to $\widetilde{M_2}$ of length less than $\frac{\varepsilon}{2} + d(M_1, M_2)$. So $\pi(\alpha)$ is of length less than $\frac{\varepsilon}{2} + d(M_1, M_2)$, which is a contradiction to our earlier hypothesis that $d(M_1, M_2) + \varepsilon = \min\{f(\partial M_1, M_2), d(\partial M_2, M_1)\}$ for some $\varepsilon > 0$. q.e.d.

Theorem 5.3 (Regular Neighborhood Theorem)**.** *Suppose M is a nonflat, properly embedded, orientable minimal surface in a complete, orientable flat three-manifold N^3 and M has absolute Gaussian curvature bounded from above by a constant $\frac{1}{C^2}$. Then the open neighborhood of M in N^3 of radius C is a regular neighborhood of M. In other words,*

the exponential map on the open C-normal interval bundle of M is injective.

Proof. This theorem is essentially a corollary of the maximum principle at infinity. We will let $N_C(M)$ denote the open C-interval bundle of M in its normal bundle. Let $\exp\colon N_C(M) \to \mathbb{R}^3$ denote the exponential map and $\widetilde{N}_C(M)$ the image. Our goal is to prove that exp is injective.

Suppose $\widetilde{M}$ is a lift of M to the universal cover $\mathbb{R}^3$ of N^3 and $\exp\colon N_C(M) \to \mathbb{R}^3$ is injective. Then $N_C(\widetilde{M})$ can be thought of as a regular neighborhood of $\widetilde{M}$ of radius C. Since M is orientable, not flat, and N^3 is orientable, $\pi_1(M)$ maps onto $\pi_1(N^3)$ (see [**9**]). It follows that $\pi_1(N^3)$, thought of as covering transformations on $\mathbb{R}^3$, leaves $\widetilde{L}$ invariant, and hence leaves $N_C(\widetilde{M})$ invariant. Therefore, the neighborhood of M in N^3 of distance C from M is the quotient $N_C(\widetilde{M})/\pi_1(N^3)$. Thus, we may assume that $N^3 = \mathbb{R}^3$.

We first check that $\exp^{-1}(M) = M_2$ where M_2 is the zero section of $N_C(M)$. Suppose this property fails to hold. Let N be a unit normal field to M. Since $\exp^{-1}(M)$ consist of components that are proper minimal surfaces, one of which is $M_2 =$ zero section, there is another proper component M_1. The proof of the maximum principle at infinity applied to M_1 and M_2 in $N_C(M)$ gives a contradiction, since the distance between these surfaces is not obtained as the distance of one to the boundary of the other one. This contradiction shows $\exp^{-1}(M) = M_2$ and, by the triangle inequality, $\exp|_{N_{\frac{C}{2}}(M)}$ is injective. This proves the existence of a regular neighborhood of M of radius half of that required in the theorem.

If $\exp\colon N_C(M) \to \mathbb{R}^3$ is not injective, then there exists ε, $0 < \varepsilon < \frac{C}{2}$, such that $\exp\colon \overline{N}_{C-\varepsilon}(M) \to \mathbb{R}^3$ is not injective. It follows that there is a $t \in (-C, -\frac{C}{2})$ or $(\frac{C}{2}, C)$ such that $\exp^{-1}(\exp(t \cdot N))$ contains a component W with $\partial W \subset \partial(\overline{N}_{C-\varepsilon}(M))$. Furthermore, W separates $\overline{N}_{C-\varepsilon}(M)$ into two regions. Let R denote the region containing M_2 and W in its boundary. Note that W is mean convex when considered to be part of the boundary of R. Replace W by a least area current S homologous to W in R with boundary ∂W. Note that $S(\delta) = S \cap \overline{N}_{C-(\varepsilon+\delta)}(M)$ is embedded and minimal for any $\delta > 0$ and nonempty for δ sufficiently small. Assume that δ is chosen sufficiently small so that $S(\delta) = \emptyset$. The existence of $S(\delta)$ and M_2 contradict the maximum principle at infinity given in Theorem 5.1. This contradiction implies $\exp\colon N_C(M) \to \mathbb{R}^3$ is injective, which completes the proof of the theorem. q.e.d.

Corollary 5.4. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with Gaussian curvature bounded from below by $-K$. Then M has area growth $A(r) \leq cr^3$, where r is the radial function in $\mathbb{R}^3$ and c is a constant that depends only on $-K$.*

Proof. Suppose K is the supremum of the absolute Gaussian curvature of M. Let $C = 1/\sqrt{K}$. Note that the area of $M(r) = M \cap B(r)$ is less than or equal to the volume of $N_{C/2}(M(r))$ times a constant. Since the volume of $B(r)$ grows cubically in r, the area of $M(r)$ must be bounded from above by cr^3 for some constant c. q.e.d.

Note that a sequence $M(i)$ of properly embedded minimal surfaces in $\mathbb{R}^3$ always has a convergent subsequence if there are uniform local bounds on the area and curvature. Hence, our next corollary is a consequence of the previous one.

Corollary 5.5. *Let $M(k_1, k_2)$ be the space of properly embedded minimal surfaces in $\mathbb{R}^3$ with the supremum of the absolute Gaussian curvature of the surface lies in $[k_1, k_2]$, where k_1, k_2 are positive numbers. Suppose these surfaces are further normalized so that every surface in $M(k_1, k_2)$ passes through the origin at a point where the absolute Gaussian curvature is at least k_1. Then $M(k_1, k_2)$ is compact in the topology of C^1-convergence on compact sets of $\mathbb{R}^3$.*

6. Surfaces of constant mean curvature and bounded Gaussian curvature

In this section we will prove results similar to the previous Theorem 2.1 and Theorem 5.3, but in the case of constant mean curvature surfaces. We begin with the following properness theorem.

Theorem 6.1. *Suppose N^3 is a complete flat three-manifold and M is a complete surface of bounded Gaussian curvature. If $f\colon M \to N^3$ is an injective immersion of constant mean curvature, then f is proper.*

Proof. By covering space arguments, we may assume $N^3 = \mathbb{R}^3$. By Theorem 2.1, the theorem is known if the mean curvature is zero. Assume now that the mean curvature is a positive constant C. As in the minimal case, the second fundamental form of M is bounded. As in the proof of Theorem 2.1, if f is not proper, then there is limit leaf[1] L of $\overline{f(M)}$ which is complete, has bounded Gaussian curvature and has constant mean curvature C. As in the proof of Theorem 2.1, we lift all questions to some ε-neighborhood $N_\varepsilon(L)$ of the normal bundle of L. We assume that ε is chosen small enough so that $N_\varepsilon(L)$ submerses into $\mathbb{R}^3$ under the exponential map, and we pull back M to a surface $\widetilde{L}(f(M))$ that limits to L. We wish to show that L is stable. Since L has nonzero mean curvature, $N_\varepsilon(L)$ is trivial and L is stable if and only if the natural projection of the universal cover is stable. Thus, as in the proof

[1]The closure of $f(M)$ has the structure of a "CMC-lamination", which consists of immersed nontransversely intersecting surfaces of constant mean curvature C. See [**18**] for an explicit definition and applications of this concept.

of Theorem 2.1, we may assume, after possibly lifting to the universal cover of $N_\varepsilon(L)$, that L and $N_\varepsilon(L)$ are both simply connected.

In order to prove that $L \subset N_\varepsilon(L)$ is stable, it is sufficient to show that for every smooth, compact, simply connected subdomain $K \subset L$, there is no function $u\colon K \to \mathbb{R}^3$, such that $\int_K u\,dA = 0$ and $S(u) = \lambda u$ for some $\lambda < 0$, where S is the stability operator for area for L. Suppose that such a K exists with a u with $S(u) + \lambda u = 0$ and $\lambda < 0$.

Consider the family of surfaces $K(t)$, $0 \le t \le \delta$, given by $\exp(tuN_K)$ where N_K is the unit normal vector field to K. The negativity of λ is equivalent to the property that for t small, the surface $K(t)$ has mean curvature greater than C at points where u is positive and mean curvature less than C where u is negative. Here we are taking the orientation of $K(t)$ induced from the orientation of K. Since L is a limit leaf of M, for $\eta > 0$ and sufficiently small, $\widetilde{M}$ intersects the η-neighborhood of K in a sequence of graphs that converge to K. It is clear that the family $K(t)$ must intersect one of these component graphs for a smallest t_0. A comparison principle of mean curvatures at such a first point of contact gives a contradiction of the usual comparison of mean curvature for surfaces that intersect in their interiors at a point where one surface lies locally on one side of the other surface. This contradiction proves that L is stable.

Since L is stable, L is a round sphere [**3**]. Since L is compact and simply connected, a standard holonomy argument implies that L cannot be a limit of a connected embedded surface in $\mathbb{R}^3$ with bounded second fundamental form; one can lift the sphere to the nearby leaves of the lamination by the holonomy so they are also spheres. Hence, we have proven that the surface M must be proper. q.e.d.

Remark 6.2. If $f\colon M \to N^3$ is a proper isometric immersion of a surface into a flat three-manifold, then for any compact ball $B \subset N^3$, $f^{-1}(B)$ is compact. It follows that $f(M) \subset N^3$ has locally bounded Gaussian curvature in N^3. The proof of Theorem 6.1 implies more generally that if $f\colon M \to N^3$ is an injective immersion of a complete, nonzero constant mean curvature surface of locally bounded Gaussian curvature, then f is proper.

We now prove that a properly embedded, constant mean curvature surface of bounded Gaussian curvature has a tubular neighborhood on its mean convex side. Our next theorem was discovered together with Antonio Ros. Also, see the paper of Meeks and Tinaglia [**18**] for some related applications of the proofs of Theorems 6.1 and 6.3 in other Riemannian three-manifolds.

Theorem 6.3. *Suppose M is a properly embedded, constant mean curvature surface of bounded Gaussian curvature in $\mathbb{R}^3$. There exists an $\varepsilon > 0$, such that the exponential map* $\exp\colon N^*_\varepsilon(M) \to \mathbb{R}^3$ *embeds,*

where $N^*_\varepsilon(M)$ are those vectors in $N_\varepsilon(M)$ on the mean convex side of M.

Proof. By Theorem 5.3, we may assume that M has nonzero mean curvature. Suppose now that no $\varepsilon > 0$ exists. First note that since M has constant mean curvature and bounded Gaussian curvature, then the second fundamental form of M is bounded. The boundedness of the second fundamental form implies that there exists an $\varepsilon > 0$, so that $\exp\colon N^*_\varepsilon(M) \to \mathbb{R}^3$ is a submersion. Assume ε is sufficiently small so that the ε-disk $D_\varepsilon(p)$ of M is a graph over its tangent space. If $\exp^{-1}(M)$ equals the zero section of $N^*_\varepsilon(M)$, then an application of the triangle inequality shows that $N^*_{\varepsilon/2}(M) \to \mathbb{R}^3$ would be an embedding, contrary to our supposition that $\exp\colon N^*_\varepsilon(M) \to \mathbb{R}^3$ is not an embedding for any $\varepsilon > 0$. Hence, arguing by contradiction, for any $\delta > 0$, $\delta < \varepsilon$, $\exp\colon N^*_\delta(M) \to \mathbb{R}^3$ satisfies $\exp^{-1}(M)$ is not the zero section of $N^*_\delta(M)$.

Let p_i, q_i be distinct pairs of points on M such that $q_i = \exp(t_i N(p_i))$, where $N(p_i)$ is the unit normal of p_i in $N^*_\varepsilon(M)$, $t_i \to 0$, t_i is the smallest positive number such that $\exp(t_i N(p_i))$ is in M and $\lim_{i\to\infty} d(p_i, q_i) = 0$. Since the fundamental form of M is bounded, then, for i large, the tangent space $T_{p_i}M$ to M at p_i is almost parallel to the tangent space to $T_{q_i}M$. Since M is proper, M separates $\mathbb{R}^3$ and so the orientations of $T_{p_i}M$ and $T_{q_i}(M)$ are opposite of each other.

Next translate M so that q_i is at the origin in $\mathbb{R}^3$. Since M has bounded second fundamental form, a subsequence of the disks $D_\varepsilon(q_i)$ converge to a disk D of constant positive mean curvature. By construction, the corresponding subsequence of the $D_\varepsilon(p_i)$ also converge to a disk D' on one side of D. However, thought of as a limit of the $D_\varepsilon(q_i)$ with orientation given as normal graphs over D, D' has negative mean curvature, which is the opposite sign it would have been from being below D. This inconsistency of sign gives a contradiction since we are in the special case where the mean curvature of M is not zero. This contradiction implies the theorem. q.e.d.

As in the minimal case, the existence of a one-sided tubular neighborhood for a properly embedded, constant mean curvature surface gives rise to area estimates.

Corollary 6.4. *Suppose M is a properly embedded surface in $\mathbb{R}^3$ of constant mean curvature. If M has bounded absolute Gaussian curvature, then M has at most cubical area growth with respect to the radial function.*

Remark 6.5. The same argument that we applied in the proof of Theorem 6.3 works to prove that any properly embedded, codimension–one submanifold of $\mathbb{R}^n$ with nonzero constant mean curvature and bounded second fundamental form has a fixed size regular neighborhood on its mean convex side.

7. A special result concerning recurrent minimal surfaces

Recall that a connected Riemannian surface is recurrent, if almost all Brownian paths on the surface are dense in the surface. Being recurrent implies, among other things, that every positive harmonic function on the surface is constant. In the case of finite topology, being recurrent is equivalent to being conformally diffeomorphic to a closed Riemann surface punctured in a finite number of points.

Theorem 7.1. *Suppose M is a surface of finite topology and one end and $f\colon M \to \mathbb{R}^3$ is a proper minimal immersion. Suppose there is a plane $P \subset \mathbb{R}^3$ such that f is transverse to P except at a finite number of points, and $f^{-1}(P)$ contains a finite number of components. Then, in the underlying conformal structure, M is conformally a closed Riemann surface punctured in a single point.*

Proof. After a rotation, we may assume that the plane P is the (x_1, x_2)-plane. In [**6**], it is shown that the surface $M(+) = \{(x_1, x_2, x_3) \in M \mid x_3 \geq 0\}$ is parabolic. We will use this parabolicity property of $M(+)$ to prove that when M has finite topology and satisfies the hypothesis in the theorem, then it is conformally a closed Riemann surface punctured in one point.

Suppose that E is an annular end representative which does not have conformal representative which is a punctured disk. Such an end is called hyperbolic and has a representative which is conformally diffeomorphic to $E = \{z \in \mathbb{C} \mid \varepsilon \leq z < 1\}$ for some positive $\varepsilon < 1$. In this conformal parametrization of E, the unit circle corresponds to points at infinity on E. After restricting to a subend of E, we may assume that $f|_E$ intersects P transversely in a finite positive number of arcs. We may also assume that each noncompact arc of the intersection has one end point on the compact boundary circle of E.

We wish to prove that each of the finite number of noncompact arcs $\alpha_1, \ldots, \alpha_n$ in $(f|_E)^{-1}(P)$ has a well-defined limit on the unit circle S^1 of points at infinity. If we can prove this, then there is an open arc $\delta \subset S^1$ which does not contain limit points of $\alpha_1, \ldots, \alpha_n$. Hence, there would be a half open disk $D \subset E$ centered at a point in δ, such that $D \cap (f^{-1}(P)) = \emptyset$, a contradiction of the fact that $M(+)$ is parabolic. (D does not have full harmonic measure but D is a proper domain which is contained in the parabolic surface $M(+)$.) Thus, to prove the theorem, we need only show that each of the arcs $\alpha_1, \ldots, \alpha_n$ has a well-defined limit point in S^1.

Suppose α_1 has two distinct limit points $p_1, p_2 \in S^1$. We first prove that at least one of the two interval components I_1, I_2 of $S^1 - \{p_1, p_2\}$ consists of limit points of α_1. Suppose not and let $s_1 \in I_1$, $s_2 \in I_2$ be points which are not limit points. Since they are not limit points, there exists a $\delta > 0$ such that the radial arcs β_1 and β_2 in E of length δ and

orthogonal to S^1 at s_1, s_2, respectively, are disjoint from α_1. Since α_1 is proper and α_1 is disjoint from $\beta_1 \cup \beta_2$, the parameterized arc $\alpha_1(t)$ must eventually be in one of the two components of $\{x \in E - (\beta_1 \cup \beta_2) \mid |x| \geq 1 - \delta\}$. Thus, α_1 cannot have both p_1 and p_2 as limit points, contrary to our hypothesis. This contradiction proves that at least one of the intervals, say I_1, consists of limit points of α_1.

Next choose a closed half disk $\overline{D} \subset \overline{E} = E \cup S^1$, centered at a point $p \in I_1$, where from above I_1 consists entirely of limit points of α_1, and suppose that the disk $\overline{D}$ is chosen sufficiently small so that $\partial\overline{D} \cap S^1 \subset I_1$. Consider the holomorphic function $g = x_3 + ix_3^*\colon E \to \mathbb{C}$, where x_3^* is the conjugate harmonic function to x_3. Note that x_3^* is well-defined on E, by Cauchy's Theorem, since the generator of the homology of E is a boundary in M. Since the plane P is transverse to E and $x_3 = 0$ on $f^{-1}(P)$, g restricted to any of the finite number of components in $f^{-1}(P) \cap E$, monotonically parameterizes an interval on the imaginary axis $\mathbb{R}(i) \subset \mathbb{C}$. Let ξ be the finite collection of these interval images in $\mathbb{R}(i)$. Thus, it is possible to find a compact interval $\beta \subset \mathbb{R}(i)$ which is disjoint from the end points of the intervals in ξ. Since $g^{-1}(\beta)$ is compact, one can choose $\overline{D}$, defined above, sufficiently small so that $\overline{D} \cap g^{-1}(\beta)$ is empty. Assume that $\overline{D}$ has been so chosen.

Let D be the interior of $\overline{D}$. Since $g|_D\colon D \to \mathbb{C} - \beta$, the function $g|_D$ is essentially bounded in the sense that it maps the disk D into a domain that is conformally equivalent to an open subset of the unit disk (via the Riemann mapping theorem). It follows from Fatou's lemma that the holomorphic function $g|_D$ has radial limits almost everywhere. Note that each limit value is a point in $\mathbb{C} \cup \{\infty\} - \beta$. Since α_1 contains the limit set $I_1 \subset \partial D$ and $g \circ \alpha_1\colon [0, \infty) \to \mathbb{R}(i)$ is monotonic, the points of I_1 with radial limits for g have a constant value which corresponds to the limiting endpoint of the curve $g \circ \alpha_1$ in $\mathbb{R}(i) \cup \{\infty\}$. But by Privalov's theorem, a nonconstant meromorphic function on the unit disk cannot have a constant radial limit on a set of ∂D with positive measure. This contradiction proves that α_1 must have a unique limit value on the circle S^1 corresponding to the points at infinity in E. As we observed earlier, this completes the proof of Theorem 7.1. q.e.d.

Remark 7.2. Every properly immersed minimal surface in $\mathbb{R}^3$ with one end can be seen to be a limit of a sequence of properly immersed minimal surfaces which are conformally hyperbolic[2] minimal surfaces of finite topology and one end (see [**1**]). It follows from Theorem 7.1 that for any plane P which intersects transversely such a properly immersed, hyperbolic finite topology surface M, the set $P \cap M$ consists of an infinite proper collection of immersed arcs of which only a finite number are closed immersed curves.

[2]A surface is *hyperbolic* if it is conformally diffeomorphic to the complement of some disk in a compact Riemann surface.

We end this paper with the following conjecture related to Theorem 7.1.

Conjecture 7.3. If $f\colon M \to \mathbb{R}^3$ a proper minimal immersion of a complete surface such that there exists a plane P that intersects M transversely, except at a finite number of points, and $f^{-1}(P)$ consists of a finite number of components, then M is recurrent.

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