Conjecture (Convex Curve Conjecture, Meeks)

*Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.*

- Results of **Meeks** and **White** indicate that the **Convex Curve Conjecture** holds in the case where the two convex planar curves lie on the boundary of their convex hull; in this case, the planar Jordan curves are called **extremal**.

- Results of **Ekholm, White** and **Wienholtz** imply every compact minimal surface that arises as a counterexample to the **Convex Curve Conjecture** is **embedded**, and that for a fixed pair of extremal, convex planar curves, there is a **bound on the genus** of such a minimal surface.

- **Meeks** has conjectured that if $\Gamma = \{\alpha, \beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{R}^3$ is a finite collection of planar, convex, simple closed curves with $\alpha$ in one plane and with $\{\beta_1, \beta_2, \ldots, \beta_n\}$ bounding a pairwise disjoint collection of disks in a parallel plane, then any compact minimal surface with boundary $\Gamma$ must have genus 0.
Conjecture (4\(\pi\)-Conjecture, Meeks, Yau, Nitsche)

If \(\Gamma\) is a simple closed curve in \(\mathbb{R}^3\) with total curvature at most 4\(\pi\), then \(\Gamma\) bounds a unique compact, orientable, branched minimal surface and this unique minimal surface is an embedded disk.

- **Nitsche** proved that a regular analytic Jordan curve in \(\mathbb{R}^3\) whose total curvature is at most 4\(\pi\) bounds a unique minimal disk.

- **Meeks** and **Yau** demonstrated the conjecture if \(\Gamma\) is a \(C^2\)-extremal curve (they even allowed the minimal surface spanned by \(\Gamma\) to be non-orientable).

- **Ekholm**, **White** and **Wienholtz** conjecture:

  Besides the unique minimal disk given by Nitsche’s Theorem, only one or two Möbius strips can occur; and if the total curvature of \(\Gamma\) is at most 3\(\pi\), then there are no such Möbius strip examples.
Conjecture (Isolated Singularities Conjecture, Gulliver, Lawson)

If $M \subset B - \{(0,0,0)\}$ is a smooth properly embedded minimal surface with $\partial M \subset \partial B$ and $\overline{M} = M \cup \{(0,0,0)\}$, then $\overline{M}$ is a smooth compact minimal surface.
Conjecture (Fundamental Singularity Conjecture, Meeks, Pérez, Ros)

If \( A \subset \mathbb{R}^3 \) is a closed set with zero 1-dimensional Hausdorff measure and \( \mathcal{L} \) is a minimal lamination of \( \mathbb{R}^3 - A \), then \( \mathcal{L} \) extends to a minimal lamination of \( \mathbb{R}^3 \).

The related **Local Removable Singularity Theorem for H-laminations** by Meeks, Pérez and Ros is a cornerstone for the proofs of:

- the **Quadratic Curvature Decay Theorem**
- the **Dynamics Theorem**
- the **Finite Topology Closure Theorem**,

which illustrate the usefulness of removable singularities results.
A minimal graph in $\mathbb{R}^3$ with zero boundary values over a proper, possibly disconnected domain in $\mathbb{R}^2$ can have at most two non-planar components. If the graph also has sublinear growth, then such a graph with no planar components is connected.

Consider a proper, possibly disconnected domain $D$ in $\mathbb{R}^2$ and a solution $u : D \rightarrow \mathbb{R}$ of the minimal surface equation with zero boundary values, such that $u$ is non-zero on each component of $D$.

- In 1981 Mikljukov proved that if each component of $D$ is simply-connected with a finite number of boundary components, then $D$ has at most three components.
- Current tools show that his method applies to the case that $D$ has finitely generated first homology group.
- Li and Wang proved that the number of disjointly supported minimal graphs with zero boundary values over an open subset of $\mathbb{R}^2$ is at most 24.
- Next Tkachev proved the number of disjointly supported minimal graphs is at most three.
In the discussion of the conjectures that follow, it is helpful to fix some notation for certain classes of complete embedded minimal surfaces in $\mathbb{R}^3$.

**Notation**

- $\mathcal{C} = \text{the space of connected, complete, embedded minimal surfaces.}$
- $\mathcal{P} \subset \mathcal{C} = \text{the subspace of properly embedded surfaces.}$
- $\mathcal{M} \subset \mathcal{P} = \text{the subspace of surfaces with more than one end.}$
Conjecture (Finite Topology Conjecture I, Hoffman, Meeks)

An orientable surface $M$ of finite topology with genus $g$ and $k$ ends, $k \neq 0, 2$, occurs as a topological type of a surface in $C$ if and only if $k \leq g + 2$.

- The method of Weber and Wolf indicates that the existence implication in the Finite Topology Conjecture holds when $k > 2$.
- Meeks, Pérez and Ros proved that for each positive genus $g$, there exists an upper bound $e(g)$ on the number of ends of an $M \in M$ with finite topology and genus $g$. Hence, the non-existence implication follows if one can show that $e(g)$ can be taken as $g + 2$.
- Concerning the case $k = 2$, the only examples in $M$ with finite topology and two ends are catenoids (Collin, Schoen, Colding-Minicozzi).
- If $M$ has finite topology, genus zero and at least two ends, then $M$ is a catenoid (Lopez, Ros).
Conjecture (Finite Topology Conjecture II, Meeks, Rosenberg)

For every non-negative integer $g$, there exists a unique non-planar $M \in \mathcal{C}$ with genus $g$ and one end.

- **Hoffman, Weber** and **Wolf** and **Hoffman** and **White** proved existence of a genus one helicoid.
- This existence proof is based on the earlier computational construction by **Hoffman, Karcher** and **Wei**.
- For genera $g = 2, 3, 4, 5, 6$, there are computational existence results.
Conjecture (Infinite Topology Conjecture, Meeks)

A non-compact, orientable surface of infinite topology occurs as a topological type of a surface in $\mathcal{P}$ if and only if it has at most one or two limit ends, and when it has exactly one limit end, then its limit end with infinite genus.

- In the infinite topology case, either $M$ has infinite genus or $M$ has an infinite number of ends.
- Such an $M$ must have at most two limit ends (Collin, Kusner, Meeks and Rosenberg).
- Such an $M$ cannot have one limit end and finite genus (Meeks, Pérez and Ros).
Conjecture (Liouville Conjecture, Meeks)

If $M \in \mathcal{P}$ and $h: M \rightarrow \mathbb{R}$ is a positive harmonic function, then $h$ is constant.

- If $M \in \mathcal{P}$ has finite genus, a limit end of genus 0 (Meeks, Perez, Ros) or two limit ends (Collin, Kusner, Meeks, Rosenberg), then $M$ is recurrent for Brownian motion, which implies $M$ satisfies the Liouville Conjecture.
- By work of Meeks, Pérez and Ros, the above conjecture holds for all of classical examples.

Below is a related conjecture.

Conjecture (Multiple-End Recurrency Conjecture, Meeks)

If $M \in \mathcal{M}$, then $M$ is recurrent for Brownian motion.
Conjecture (Isometry Conjecture, Choi, Meeks, White)

If \( M \in \mathcal{C} \), then every intrinsic isometry of \( M \) extends to an ambient isometry of \( \mathbb{R}^3 \). Furthermore, if \( M \) is not a helicoid, then it is minimally rigid, in the sense that any isometric minimal immersion of \( M \) into \( \mathbb{R}^3 \) is congruent to \( M \).

The Isometry Conjecture is known to hold if:

- \( M \in \mathcal{M} \) (Choi, Meeks and White),
- \( M \) is doubly-periodic (Meeks and Rosenberg),
- \( M \) is periodic with finite topology quotient (Meeks and Pérez),
- \( M \) has finite genus (Meeks and Tinaglia).

One can reduce the Isometry Conjecture to checking that whenever \( M \in \mathcal{P} \) has one end and infinite genus, then there exists a plane in \( \mathbb{R}^3 \) that intersects \( M \) in a set that contains a simple closed curve. The reason for this reduction is that the flux of \( M \) along such a simple closed curve is not zero, and hence, none of the associate surfaces to \( M \) are well-defined; but Calabi proved that the associate surfaces are the only isometric minimal immersions from \( M \) into \( \mathbb{R}^3 \), up to congruence.
**Conjecture (Scherk Uniqueness Conjecture, Meeks, Wolf)**

If $M$ is a connected, properly immersed minimal surface in $\mathbb{R}^3$ and $\text{Area}(M \cap B(R)) \leq 2\pi R^2$ holds in balls $B(R)$ of radius $R$, then $M$ is a plane, a catenoid or one of the singly-periodic Scherk minimal surfaces.

- By the Monotonicity Formula, any connected non-flat, properly immersed minimal surface in $\mathbb{R}^3$ with
  \[
  \lim_{R \to \infty} R^{-2} \text{Area}(M \cap B(R)) \leq 2\pi,
  \]
  is embedded.

- **Meeks** and **Wolf** proved the **Scherk Uniqueness Conjecture** holds under the assumption that the surface is periodic.
Conjecture (Unique Limit Tangent Cone Conjecture, Meeks)

If $M \in \mathcal{P}$ is not a plane and has quadratic area growth, then

$$\lim_{t \to \infty} \frac{1}{t} M$$

exists and is a minimal, possibly non-smooth cone over a finite balanced configuration of geodesic arcs in the unit sphere, with common ends points and integer multiplicities.

Meeks and Wolf’s proof of the Scherk Uniqueness Conjecture in the periodic case uses that the Unique Limit Tangent Cone Conjecture above holds in the periodic setting; this approach suggests to solve the Scherk Uniqueness Conjecture by:

- First to prove the uniqueness of the limit tangent cone of $M$, from which it should follow that $M$ has two Alexandrov-type planes of symmetry.
- Next use these planes of symmetry to describe the Weierstrass representation of $M$. Meeks and Wolf claim this would be sufficient to complete the proof of the conjecture.
Conjecture (Injectivity Radius Growth Conjecture, Meeks, Pérez, Ros)

An $\mathbf{M} \in \mathcal{C}$ has finite topology if and only if its injectivity radius function grows at least linearly with respect to the extrinsic distance from the origin.

If $\mathbf{M} \in \mathcal{C}$ has finite topology, then $\mathbf{M}$ has finite total curvature or is asymptotic to a helicoid. So there exists a constant $C_M > 0$ such that the injectivity radius function $I_M : \mathbf{M} \rightarrow (0, \infty]$ satisfies

$$I_M(p) \leq C_M \|p\|, \quad p \in \mathbf{M}.$$  

Work Meeks, Pérez and Ros indicates that this linear growth property of the injectivity radius function characterizes the finite topology examples in $\mathcal{C}$. 
Conjecture (Negative Curvature Conjecture, Meeks, Pérez, Ros)

If $M \in C$ has negative curvature, then $M$ is a catenoid, a helicoid or one of the singly or doubly-periodic Scherk minimal surfaces.

- Suppose $M \in C$ has finite topology. $M$ either has finite total curvature or is a helicoid with handles. Such a surface has negative curvature if and only if it is a catenoid or a helicoid.
- Suppose $M \in C$ is invariant under a proper discontinuous group $G$ of isometries of $\mathbb{R}^3$ and $M/G$ has finite topology. Then $M/G$ is properly embedded in $\mathbb{R}^3/G$ (Meeks, Pérez, Ros) and $M/G$ has finite total curvature (Meeks, Rosenberg). If $M/G$ has negative curvature and the ends of $M/G$ are helicoidal or planar, then $M$ is easily proven to have genus zero, and so, it is a helicoid. If $M/G$ is doubly-periodic, then $M$ is a Scherk minimal surface. In the case $M/G$ is singly-periodic, then $M$ must have Scherk-type ends but we do not know if the surface must be a Scherk singly-periodic minimal surface.
Conjecture (Four Point Conjecture, Meeks, Pérez, Ros)

Suppose $M \in C$. Then:

1. If the Gauss map of $M$ omits 4 points on $S^2(1)$, then $M$ is a singly or doubly-periodic Scherk minimal surface.

2. If the Gauss map of $M$ omits exactly 3 points on $S^2(1)$, then $M$ is a singly-periodic **Karcher saddle tower** whose flux polygon is a convex unitary hexagon. (note that any three points in a great circle are omitted by one of these examples).

3. If the Gauss map of $M$ omits exactly 2 points, then $M$ is a catenoid, a helicoid, one of the Riemann minimal examples or one of the **KMR doubly-periodic minimal tori**. In particular, the pair of points missed by the Gauss map of $M$ must be antipodal.
Conjecture (Four Point Conjecture, Meeks, Pérez, Ros)

Suppose $M \in \mathcal{C}$. Then:

1. If the Gauss map of $M$ omits 4 points on $S^2(1)$, then $M$ is a singly or doubly-periodic Scherk minimal surface.

2. If the Gauss map of $M$ omits exactly 3 points on $S^2(1)$, then $M$ is a singly-periodic Karcher saddle tower whose flux polygon is a convex unitary hexagon. (note that any three points in a great circle are omitted by one of these examples).

3. If the Gauss map of $M$ omits exactly 2 points, then $M$ is a catenoid, a helicoid, one of the Riemann minimal examples or one of the KMR doubly-periodic minimal tori. In particular, the pair of points missed by the Gauss map of $M$ must be antipodal.

- A classical result of Fujimoto is that the Gauss map of any orientable, complete, non-flat, minimally immersed surface in $\mathbb{R}^3$ cannot exclude more than 4 points.

- If one assumes that a surface $M \in \mathcal{C}$ is periodic with finite topology quotient, then Meeks, Pérez and Ros have solved the first item in the above conjecture.
Conjecture (Finite Genus Properness Conjecture, Meeks, Pérez, Ros)

If $M \in C$ and $M$ has finite genus, then $M \in P$.

- **Colding** and **Minicozzi** proved the conjecture for surfaces of **finite topology**.
- **Meeks**, **Pérez** and **Ros** proved the **Finite Genus Properness Conjecture** under the additional hypothesis that $M$ has a **countable number of ends** or even a countable number of limit ends.
- **Meeks**, **Pérez** and **Ros** had conjectured that if $M \in C$ has finite genus, then $M$ has bounded Gaussian curvature, which they proved is equivalent to the above conjecture.
Let $M$ be open surface.

1. There exists a complete proper minimal embedding of $M$ in every smooth bounded domain $D \subset \mathbb{R}^3$ iff $M$ is orientable and every end has infinite genus.

2. There exists a complete proper minimal embedding of $M$ in some smooth bounded domain $D \subset \mathbb{R}^3$ iff every end of $M$ has infinite genus and $M$ has a finite number of nonorientable ends.

3. There exists a complete proper minimal embedding of $M$ in some particular non-smooth bounded domain $D \subset \mathbb{R}^3$ iff every end of $M$ has infinite genus.
Conjecture (Embedded Calabi-Yau Conjectures)

- There exists an $M \in \mathcal{C}$ whose closure $\overline{M}$ has the structure of a minimal lamination of a slab, with $M$ as a leaf and with two planes as limit leaves. In particular, $\mathcal{P} \neq \mathcal{C}$ (Meeks).
- A connected, complete, embedded surface of non-zero constant mean curvature in $\mathbb{R}^3$ with finite genus is properly embedded (Meeks, Tinaglia).
1. A complete, non-orientable, stable minimal surface in $\mathbb{R}^3$ with compact boundary has finite total curvature (Ros).

2. If $A \subset \mathbb{R}^3$ is a closed set with zero 1-dimensional Hausdorff measure and $M \subset \mathbb{R}^3 - A$ is a connected, stable, minimally immersed surface which is complete outside of $A$, then the closure of $M$ is a plane (Meeks).

3. If $M \subset \mathbb{R}^3$ is a minimal graph over a proper domain in $\mathbb{R}^2$ with boundary, then $M$ is parabolic (López, Meeks, Pérez, Weitsman).

4. If $M \subset \mathbb{R}^3$ is a complete, stable minimal surface with boundary, then $M$ is $\delta$-parabolic (Meeks, Rosenberg).

5. A complete, embedded, stable minimal surface in $\mathbb{R}^3$ with boundary a straight line is a half-plane, a half of the Enneper minimal surface or a half of the helicoid$^a$(Pérez, Ros, White).

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$^a$Pérez has proved this conjecture under the additional assumption that the surface is proper and has quadratic area growth.
Since if some flux vector of a minimally immersed $M$ in $\mathbb{R}^3$ is non-zero, then the inclusion map is the unique isometric minimal immersion of $M$ into $\mathbb{R}^3$ up to congruence, the One-Flux Conjecture below implies the Isometry Conjecture.

**Conjecture (One-Flux Conjecture, Meeks, Pérez, Ros)**

Let $M \in \mathcal{C}$ and let

$$\mathcal{F}_M = \{ F(\gamma) = \int_\gamma \text{Rot}_{90^\circ}(\gamma') \mid \gamma \in H_1(M, \mathbb{Z}) \}$$

be the abelian group of flux vectors for $M$. If $\mathcal{F}_M$ has rank at most 1, then $M$ is a plane, a helicoid, catenoid, a Riemann minimal example or a doubly-periodic Scherk minimal surface.
Conjecture (Standard Middle End Conjecture, Meeks)

If $M \in \mathcal{M}$ and $E \subset M$ is a one-ended representative for a middle end of $M$, then $E$ is $C^0$-asymptotic to the end of a plane or catenoid. In particular, if $M$ has two limit ends, then each middle end is $C^0$-asymptotic to a plane.

The above conjecture can be viewed as a generalization of the fact:

An **annular end** $E$ of a surface $M \in \mathcal{M}$ with more than one end is $C^2$-asymptotic to the end of a plane or catenoid.
An end \( e \in \mathcal{E}(M') \) of a non-compact Riemannian manifold \( M' \) is called **massive** or **non-parabolic** if every open, proper subdomain \( \Omega \subset M' \) with compact boundary that represents \( e \) is massive (i.e. there exists a bounded subharmonic function \( v : M' \to [0, \infty) \) such that \( v = 0 \) in \( M' - \Omega \) and \( \sup_{\Omega} v > 0 \).)

**Conjecture (Meeks, Pérez, Ros)**

Let \( M \in \mathcal{M} \) with horizontal limit tangent plane at infinity. Then:

- \( M \) has a massive end if and only if it admits a non-constant, positive harmonic function (**Massive End Conjecture**).

- Any proper, one-ended representative \( E \) with compact boundary for a middle end of \( M \) has vertical flux (**Middle End Flux Conjecture**).

- Suppose that there exists a half-catenoid \( C \) with negative logarithmic growth in \( \mathbb{R}^3 - M \). Then, any proper subdomain of \( M \) that only represents ends that lie below \( C \) (in the sense of the Ordering Theorem) has quadratic area growth (**Quadratic Area Growth Conjecture**).
Riemann minimal examples near helicoid limits
Conjecture (Parking Garage Structure Conjecture, Meeks, Pérez, Ros)

Suppose $M_n \subset B(R_n)$ is a locally simply-connected sequence of embedded minimal surfaces with $\partial M_n \subset \partial B(R_n)$ and $R_n \to \infty$ as $n \to \infty$. Assume also that the sequence $M_n$ does not have uniformly bounded curvature in $B(1)$. Then:

- After a rotation and choosing a subsequence, the $M_n$ converge to a minimal parking garage structure on $R^3$ consisting of the foliation $\mathcal{L}$ of $R^3$ by horizontal planes, with singular set of convergence being a locally finite collection $S(\mathcal{L})$ of vertical lines which are the columns of the parking garage structure.

- For any two points $p, q \in R^3 - S(\mathcal{L})$, the ratio of the vertical spacing between consecutive sheets of the double multigraphs defined by $M_n$ near $p$ and $q$, converges to one as $n \to \infty$. Equivalently, the Gaussian curvature of $M_n$ blows up at the same rate along all the columns as $n \to \infty$. 