The CMC Dynamics Theorem deals with describing all of the periodic or repeated geometric behavior of a properly embedded CMC surface with bounded second fundamental form in $\mathbb{R}^3$ in order to better understand general properties that hold for all such surfaces. Today I will be discussing my joint work with Giuseppe Tinaglia at the University of Notre Dame, South Bend, Indiana, concerning the CMC Dynamics Theorem with a focus on the CMC Minimal Element Theorem.
The space $T(M)$ of translational limits of $M$

### Notation

- **$M \subset \mathbb{R}^3$** is a properly embedded CMC surface with bounded second fundamental form.
- **$W_M$** is the closed connected component in $\mathbb{R}^3$ on the mean convex side of $M$.
- **$L(M)$** is the set of all properly immersed (not necessarily connected) surfaces $\Sigma \subset \mathbb{R}^3$ which are limits of some sequence of translates $M - p_n$, where $p_n \in M$ with $|p_n| \to \infty$.
- **$T(M)$** is the set of (pointed) components of surfaces in $L(M)$ passing through the origin.
On the left is the singly-periodic surface $M$, which is the CMC desingularization of the collection of singly-periodic spheres on the right.

Elements of $L(M)$ are all translates of $M$ and a doubly periodic family of Delaunay surfaces which contain $\vec{0}$.

Elements of $T(M)$ are translates of $M$ passing through $\vec{0}$ and translates of a fixed Delaunay surface $D$ passing through $\vec{0}$. 


The only nonempty minimal $T$-invariant $\Delta \subset T(M)$ is $T(D)$, where $D \in T(M)$ is a fixed Delaunay surface.
Lemma

A nonempty set $\Delta \subset T(M)$ is a minimal $T$-invariant set if and only if whenever $\Sigma \in \Delta$, then $T(\Sigma) = \Delta$. 
Theorem (CMC Dynamics Theorem in homogeneous manifolds)

Let $M$ denote a noncompact, properly embedded, separating CMC hypersurface with bounded second fundamental form in a homogeneous manifold $N$. Fix a base point $p \in N$ and a transitive group $G$ of isometries. Let $T_G(M)$ the set of connected, properly immersed submanifolds passing through $p$ which are limits of a divergent sequence of compact domains on $M$ "translated" by elements in $G$. Then:

- $M$ has a fixed size regular neighborhood on its mean convex side.
- For each $\Sigma \in T_G(M) \cup \{M\}$, we have $T_G(\Sigma) \neq \emptyset$ and $T_G(\Sigma) \subset T_G(M)$.
- $T_G(M)$ and has a natural compact topological space structure induced by a metric.
- Every nonempty $T_G$-invariant subset of $T_G(M)$ contains a nonempty minimal $T_G$-invariant subset.
Key properties of minimal elements

Theorem (Minimal Element Theorem)

Suppose that $M$ has possibly nonempty compact boundary and $\Sigma \in T(M)$ is a minimal element. Then:

- $T(\Sigma) = L(\Sigma)$, i.e., every surface in $L(\Sigma)$ is connected.
- If $\Sigma$ has at least 2 ends, then $\Sigma$ is a Delaunay surface.
- $\Sigma$ is chord-arc, i.e., there exists a $c > 0$ such that for $p, q \in \Sigma$ with $d_{R^3}(p, q) \geq 1$, then
  $$d_\Sigma(p, q) \leq c \cdot d_{R^3}(p, q).$$
- For all $c, D > 0$, there exists a $d_{c, D} > 0$ such that: For every compact set $X \subset \Sigma$ with extrinsic diameter less than $D$ and for each $q \in \Sigma$, there exists a smooth compact, domain $X_{q,c} \subset \Sigma$ and a vector, $v[q, c, D] \in R^3$, so that
  $$d_\Sigma(q, X_{q,c}) < d_{c,D} \quad \text{and} \quad d_H(X, X_{q,c} + v[q, c, D]) < c.$$
The Alexandrov reflection principle at infinity

**Theorem (Halfspace Theorem, R-R, M-T)**

If \( M \subset \{ x_3 > 0 \} \), then \( T(M) \) has a minimal element with the \((x_1, x_2)\)-plane \( P \) as a plane of Alexandrov symmetry.

**Idea of the proof.**

Using the fixed sized regular neighborhood of \( M \) and the Alexandrov reflection principle, one finds a positive number \( C \) so that \( M \cap \{ x_3 < C \} \) is a graph a smooth function on some domain in \( P \) and points \( p_n \in M \cap \{ x_3 = C \} \) such that the tangent spaces to \( M \) at the points \( p_n \) converge to the vertical. A subsequence of the translated surfaces \( M - p_n \) gives rise to a limit surface \( \Sigma \in T(M) \) with the plane \( P \) as a plane of Alexandrov symmetry. By the Dynamics Theorem, \( T(\Sigma) \) contains the desired minimal element.
Lemma (Large Balls Lemma)

If $\mathbb{R}^3 - M$ contains balls of arbitrarily large radius, then $T(M)$ has a minimal element with a plane of Alexandrov symmetry.

Proof.

Find a sequence $B_n$ of such open balls so that there exist a divergent sequence of points $p_n \in M \cap \partial B_n$ and a related limit $\Sigma \in T(M)$ arising from $M - p_n$, which lies in the halfspace $\lim_{n \to \infty} (B_n - p_n) \subset \mathbb{R}^3$. Then apply the Halfspace Theorem to $\Sigma$.

Corollary

If $T(M)$ does not contain a minimal element with a plane of Alexandrov symmetry, then there is an integer $K$ such that the number of ends of $M$ or of any $\Sigma \in L(M)$ is at most $K$. 
Idea of the proof of the corollary.

Suppose that $\mathbf{T}(\mathbf{M})$ contains no minimal examples with a plane of Alexandrov symmetry. The proof uses the following fact, for any $R > 0$. Suppose $E_1, E_2, \ldots, E_k$ are disjoint end representatives for a surface $\Sigma \in \mathbf{T}(\mathbf{M})$ with boundaries in some ball $B(R - 1)$. When $k$ is sufficiently large, then for every ball $B$ of radius $R$ in $\mathbb{R}^3 - (\Sigma \cup B(R))$, $B$ is disjoint from one of these end representatives. Otherwise, one contradicts the uniform cubical volume estimate for all surfaces in $\mathbf{T}(\mathbf{M})$ in balls of radius $R$.

The proof of Large Balls Lemma now works.
Theorem (Annular End Theorem)

Suppose $M$ has a plane of Alexandrov symmetry and at least $n > 1$ ends. Then $M$ has at least $n$ annular ends.

Corollary

If $\Sigma \in T(M)$ is a minimal element, then each surface in $L(\Sigma)$ has at most one end or else $\Sigma$ is a Delaunay surface.

Proof of the corollary.

If a surface in $T(\Sigma)$ has a plane of Alexandrov symmetry, then so does $\Sigma$ and every surface in $L(\Sigma)$, and the corollary follows from the theorem. So assume that no surface in $T(\Sigma)$ has a plane of Alexandrov symmetry. If some surface $\Sigma' \in L(\Sigma)$ has $n > 1$ ends, then the Large Balls Lemma implies every surface in $L(\Sigma')$ has at least $n$ components. Choose $F \in L(\Sigma')$ with $\Sigma$ as a component. Repeating this argument, $L(F) \subset L(\Sigma')$ has an element with $2n - 1$ ends. So $T(\Sigma)$ has an element with a plane of Alexandrov symmetry, a contradiction.
Minimal elements $\Sigma \in T(M)$ are chord-arc.

**Theorem**

Minimal elements $\Sigma \in T(M)$ are chord-arc.

**Proof:** For $p, q \in \mathbb{R}^3$, $d(p, q) = d_{\mathbb{R}^3}(p, q)$. Let $\Sigma \in T(M)$ be a minimal element.

**Assertion**

There exists a function $f : [1, \infty) \rightarrow [1, \infty)$ so that for $p, q \in \Sigma$ with $1 \leq d(p, q) \leq R$, $d_{\Sigma}(p, q) \leq f(R) \cdot d(p, q)$.

**Proof.**

Otherwise there exists an $R_0$ and points $p_n, q_n \in \Sigma$ with $d(p_n, q_n) \leq R_0$ and $n \leq d_{\Sigma}(p_n, q_n)$. Then $(\Sigma - p_n) \rightarrow \Sigma_\infty \in L(\Sigma)$ which is disconnected; this contradicts previous corollary, so $f$ exists.
There exists a function $f: [1, \infty) \to [1, \infty)$ so that for $p, q \in \Sigma$ with $1 \leq d(p, q) \leq R$, $d_{\Sigma}(p, q) \leq f(R) \cdot d(p, q)$.

**Case A:** Every ball of a fixed radius $R - 1$ in $\mathbb{R}^3$ intersects $\Sigma$.

**Proof.**

Let $p, q \in \Sigma$ such that $d(p, q) \geq 4R$. Let $B_1, \ldots, B_n$ be a chain of closed balls of radius $R$ centered along the line segment joining $p, q$ and with points $s_i \in B_i \cap \Sigma$ and $s_1 = p, s_n = q$, and so that, $1 \leq d(s_i, s_{i+1}) \leq 4R$. Note $(n - 1)2R \leq d(p, q)$. 
Minimal elements $\Sigma \in T(M)$ are chord-arc

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$$d_\Sigma(p, q) \leq \sum_{i=1}^{n-1} d_\Sigma(s_i, s_{i+1}) \leq \sum_{i=1}^{n-1} f(4R)d(s_i, s_{i+1}) \leq f(4R) \cdot (n - 1)4R \leq 2f(4R) \cdot d(p, q).$$
Case B: Σ has a plane of Alexandrov symmetry. The proof of this case uses similar arguments as in Case A. This completes the proof of the chord-arc property of minimal elements.

Theorem (Annular End Theorem)
Suppose \( M \) has a plane of Alexandrov symmetry and at least \( n > 1 \) ends. Then \( M \) has at least \( n \) annular ends. In particular, \( M \) has a finite number of ends greater than 1 if and only if it has finite topology.

Proof: Suppose \( M \) is a bigraph over a domain \( \Delta \) in the \( x_1x_2 \)-plane and \( M_1, M_2 \subset M \) are ends of \( M \), which are components in the complement of a vertical cylinder of radius \( R_0 \). Suppose \( M_i \) is a bigraph over \( \Delta_i \subset \Delta \).
Figure: $\sigma_1(1)$ is the short arc in the circle of radius $R_1$. $P_1(1)$ is the yellow shaded region containing $\sigma_1(1)$ and an arc of $\partial_1$ in its boundary. By the Alexandrov reflection principle and height estimates for CMC graphs, $P_1$ lies $1/H$ close to any vertical halfspace containing $\sigma_1(1)$.

After a horizontal translation and a rotation of $M_1$ around the $x_3$-axis, we may assume that $M_1$ lies in $\{(x_1, x_2, x_3) \mid x_2 > 0\}$. The proof of the Halfspace Theorem shows that after another rotation, we may also assume $\Delta_1$ also contains divergent sequence of points $p_n = (x_1(n), x_2(n), 0) \in \partial \Delta_1$ such that $\frac{x_2(n)}{x_1(n)} \to 0$ as $n \to \infty$ and the surfaces $M_1 - p_n$ converge to a Delaunay surface.
Figure: Choosing the points $p_n \in M_1$ and related data.

Our goal is to show $M_1$ contains an annular end. This follows from the next assertion.

**Assertion**

The regions between forming Delaunay surfaces near $p_n$ are annuli.

The assertion holds if the segment $a(n) \cap \Delta_1$ bounds a compact domain in (above) $\Delta_1$. 
Existence of green bubble implies that for some $c > 0$, the CMC flux $F$ of $E_2 = \nabla x_2$ on the portion $X_n$ of $M_1$ over the shaded rectangle satisfies $F > c$, contradicting a standard application of the divergence theorem.
Figure: A picture of $M_1$ with two bubbles blown on its mean convex side.