Practice problems from old exams for math 233

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Disclaimer: Your instructor covers far more materials that we can possibly fit into a four/five questions exams. These practice tests are meant to give you an idea of the kind and varieties of questions that were asked within the time limit of that particular tests. In addition, the scope, length and format of these old exams might change from year to year. Users beware! These are NOT templates upon which future exams are based, so don't expect your exam to contain problems exactly like the ones presented here. Check the course web page for an update on the material to be covered on each exam or ask your instructor.

1 Practice problems for Final Exam.

Fall 2008 Exam

1. (a) Consider the points
   $A = (1, 0, 0)$, $B = (2, 1, 0)$ and $C = (1, 2, 3)$. Find the **parametric equations** for the line $\mathbf{L}$ passing through the points A and C.

   (b) Find an equation of the plane in $\mathbb{R}^3$ which contains the points A, B, C.

   (c) Find the area of the triangle $\mathbf{T}$ with vertices A, B and C.

2. Find the volume under the graph of $f(x, y) = x + 2xy$ and over the bounded region in the first quadrant $\{(x, y) \mid x \geq 0, \ \ y \geq 0\}$ bounded by the curve $y = -x^2 + 1$ and the x and y-axes.

3. Let
$$\mathbf{I} = \int_0^1 \int_{2x}^2 \sin(y^2)\, dy\, dx.$$

   (a) Sketch the region of integration.

   (b) Write the integral $\mathbf{I}$ with the order of integration reversed.

   (c) Evaluate the integral $\mathbf{I}$. Show your work.

4. Consider the function $\mathbf{F}(x, y, z) = x^2 + xy^2 + z$.

   (a) What is the gradient $\nabla\mathbf{F}(x, y, z)$ of $\mathbf{F}$ at the point $(1, 2, -1)$?

   (b) Calculate the directional derivative of $\mathbf{F}$ at the point $(1, 2, -1)$ in the direction of the vector $\langle 1, 1, 1\rangle$?

   (c) What is the maximal rate of change of $\mathbf{F}$ at the point $(1, 2, -1)$?

(d) Find the equation of the tangent plane to the level surface $\mathbf{F}(x, y, z) = 4$ at the point $(1, 2, -1)$.

5. Find the volume $\mathbf{V}$ of the solid under the surface $z = 1 - x^2 - y^2$ and above the xy-plane.

6. (a) Determine whether the following vector fields are **conservative** or not. Find a **potential function** for those which are indeed **conservative**.

   i. $\mathbf{F}(x, y) = \langle x^2 + e^x + xy, xy - \sin(y)\rangle$.

   ii. $\mathbf{G}(x, y) = \langle 3x^2y + e^x + y^2, x^3 + 2xy + 3y^2\rangle$.

7. Evaluate the line integral

$$\int_C yz\ dx + xz\ dy + xy\ dz,$$

where C is the curve starting at $(0, 0, 0)$, traveling along a line segment to $(1, 0, 0)$ and then traveling along a second line segment to $(1, 2, 1)$.

8. (a) Use Green's Theorem to show that if $\mathbf{D} \subset \mathbb{R}^2$ is the bounded region with boundary a positively oriented simple closed curve C, then the area of $\mathbf{D}$ can be calculated by the formula:

$$\textbf{Area} = \frac{1}{2}\int_C -y\ dx + x\ dy$$

   (b) Consider the ellipse $4x^2 + y^2 = 1$. Use the above area formula to calculate the area of the region $\mathbf{D} \subset \mathbb{R}^2$ with boundary this ellipse. (Hint: This ellipse can be parameterized by $\mathbf{r}(t) = \langle \frac{1}{2}\cos(t), \sin(t)\rangle$ for $0 \le t \le 2\pi$.)

Spring 2008 Exam

9. Use the space curve $\vec{r}(t) = \langle t^2 - 1, t^2, t/2\rangle$ for parts (a) and (b)below.

   (a) Find the velocity, speed, and acceleration of a particle whose positions function is $\vec{r}(t)$ at time $t = 4$.

   (b) Find all points where the particle with position vector $\vec{r}(t)$ intersects the plane $x + y - 2z = 0$.

10. Let D be the region of the xy plane above the graph of $y = x^2$ and below the line $y = x$.

   (a) Determine an iterated integral expression for the double integral

$$\int_D xy\, dA$$

(b) Find an equivalent iterated integral to the one found in part (a) with the reversed order of integration.

(c) Evaluate one of the two iterated integrals in parts (a,b).

11. Find the absolute maximum and absolute minimum values of $f(x, y) = x^2 + 2y^2 - 2y$ in the set $D = \{(x, y) : \ x^2 + y^2 \leq 4\}$.

12. Let D be the region in the first quadrant $x, y \geq 0$ that lies between the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

    (a) Describe the region D using polar coordinates.

    (b) Evaluate the double integral

$$\int\int_D 3x + 3y \, dA.$$

13. (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point$(1, 0, -1)$ for the implicit function z determined by the equation

$$x^3 + y^3 + z^3 - 3xyz = 0.$$

    (b) Is the tangent plane to the surface

$$x^3 + y^3 + z^3 - 3xyz = 0$$

    at the point $(1, 0, -1)$ perpendicular to the plane $2x + y - 3z = 2$? Justify your answer with an appropriate calculation.

14. (a) Consider the vector field $\mathbf{G}(x, y) = \langle 4x^3 + 2xy, x^2 \rangle$. Show that $\mathbf{G}$ is *conservative* (i.e. $\mathbf{G}$ is a *potential* or a *gradient* vector field), and use the fundamental theorem for line integrals to determine the value of

$$\int_C \mathbf{G} \cdot \mathbf{dr},$$

    where C is the contour consisting of the line starting at (2,-2) and ending at $(-1, 1)$.

    (b) Now let T denote the closed contour consisting of the triangle with vertices at (0,0),(1,0), and (1,1) with the counterclockwise orientation, and let $\mathbf{F}(x, y) = \langle \frac{1}{2}y^2 - y, xy \rangle$. Compute

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

    directly (from the definition of line integral).

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    (c) Explain how Green's theorem can be used to show that the integral in **(b)** must be equal to the area of the region D interior to T.

Fall 2007 Exam

15. Let

$$I = \int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy.$$

   (a) Sketch the region of integration
   (b) Write the integral I with the order of integration reversed.
   (c) Evaluate the integral I. Show your work.

16. Find the distance from the point $(3, 2, -7)$ to the line

$$x = 1 + t, \quad y = 2 - t, \quad z = 1 + 3t.$$

17. (a) Find the velocity and acceleration of a particle moving along the curve

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

   at the point $(2, 4, 8)$.

   (b) Find all points where the curve in part (a) intersects the surface $z = 3x^3 + xy - x$.

18. Find the volume of the solid which lies below the sphere $x^2+y^2+z^2 = 4$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 3$.

19. Consider the line integral

$$\int_C \sqrt{1+x}\, dx + 2xy\, dy,$$

where C is the triangular path starting from $(0, 0)$, to $(2, 0)$, to $(2, 3)$, and back to $(0, 0)$. This problem continues on the next page.

   (a) Evaluate this line integral directly, **without** using Green's Theorem.
   (b) Evaluate this line integral using Green's theorem.

20. Consider the vector field $\mathbf{F} = (y^2/x^2)\mathbf{i} - (2y/x)\mathbf{j}$.

   (a) Find a function f such that $\nabla f = \mathbf{F}$.
   (b) Let C be the curve given by $\mathbf{r}(t) = \langle t^3, \sin t\rangle$ for $\frac{\pi}{2} \le t \le \pi$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Fall 2006 Exam

21. Find parametric equations for the line in which the planes $x-2y+z = 1$ and $2x + y + z = 1$ intersect.

22. Consider the surface $x^2 + y^2 - 2z^2 = 0$ and the point $P(1, 1, 1)$ which lies on the surface.

    (i) Find the equation of the tangent plane to the surface at P.

    (ii) Find the equation of the normal line to the surface at P.

23. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 2x$$

on the disc $x^2 + y^2 \leq 4$.

24. Evaluate the iterated integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2 + y^2}\, dy\, dx.$$

25. Find the volume of the solid under the surface $z = 4 - x^2 - y^2$ and above the region in the xy plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

26. Determine whether the following vector fields are conservative or not. Find a potential function for those which are indeed conservative.

    (a) $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (xy - y^2)\mathbf{j}$.

    (b) $\mathbf{F}(x, y) = (3x^2y + y^2)\mathbf{i} + (x^3 + 2xy + 3y^2)\mathbf{j}$.

27. Evaluate the line integral

$$\int_C (x^2 + y)\, dx + (xy + 1)\, dy$$

where C is the curve starting at $(0, 0)$, traveling along a line segment to $(1, 2)$ and then traveling along a second line segment to $(0, 3)$.

28. Use Green's Theorem to evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = \langle y^3 + \sin 2x, 2xy^2 + \cos y\rangle$ and C is the unit circle $x^2 + y^2 = 1$ which is oriented counterclockwise.

These problems are from older exams

29. (a) Express the double integral

$$\int\int_R x^2y - x\, dA$$

as an iterated integral and evaluate it, where R is the first quadrant region enclosed by the curves $y = 0$, $y = x^2$ and $y = 2 - x$.

(b) Find an equivalent iterated integral expression for the double integral in 29a, where the order of integration is reserved from the order used in part 29a. (Do **not** evaluate this integral.)

30. Calculate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F}(x,y) = y^2x\mathbf{i} + xy\mathbf{j}$, and C is the path starting at $(1,2)$, moving along a line segment to $(3,0)$ and then moving along a second line segment to $(0,1)$.

31. Evaluate the integral

$$\int\int_R y\sqrt{x^2+y^2}\,dA$$

with R the region $\{(x,y)\colon 1 \leq x^2+y^2 \leq 2,\ 0 \leq y \leq x.\}$

32. (a) Show that the vector field $\mathbf{F}(x,y) = \left\langle \frac{1}{y} + 2x, -\frac{x}{y^2} + 1 \right\rangle$ is conservative by finding a potential function $f(x,y)$.

(b) Let C be the path described by the parametric curve $\mathbf{r}(t) = \langle 1 + 2t, 1+t^2\rangle$ for 32a to determine the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

33. (a) Find the equation of the tangent plane at the point $P = (1,1-1)$ in the level surface $f(x,y,z) = 3x^2 + xyz + z^2 = 1$.

(b) Find the directional derivative of the function $f(x,y,z)$ at $P = (1,1,-1)$ in the direction of the tangent vector to the space curve $\mathbf{r}(t) = \langle 2t^2 - t, t^{-2}, t^2 - 2t^3\rangle$ at $t = 1$.

34. Find the absolute maxima and minima of the function

$$f(x,y) = x^2 - 2xy + 2y^2 - 2y.$$

in the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 7$.

35. Consider the function $f(x,y) = xe^{xy}$. Let P be the point $(1,0)$.

(a) Find the rate of change of the function f at the point P in the direction of the point $(3,2)$.

(b) Give a direction in terms of a unit vector (there are two possibilities) for which the rate of change of f at P in that direction is zero.

36. (a) Find the work done be the vector field $\mathbf{F}(x,y) = \langle x-y, x\rangle$ over the circle $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \leq t \leq 2\pi$.

(b) Use Green's Theorem to calculate the line integral $\int_C(-y^2)\,dx + xy\,dy$, over the **positively** (counterclockwise) oriented closed curve defined by $x = 1$, $y = 1$ and the coordinate axes.

37. (a) Show that the vector field $\mathbf{F}(x,y) = \langle x^2y, \frac{1}{3}x^3\rangle$ is conservative and find a function f such that $\mathbf{F} = \nabla f$.

(b) Using the result in part **a**, calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, along the curve C which is the arc of $y = x^4$ from $(0,0)$ to $(2,16)$.

38. Consider the surface $x^2 + y^2 - \frac{1}{4}z^2 = 0$ and the point $P(1, 2, -2\sqrt{5})$ which lies on the surface.

    (a) Find the equation of the tangent plane to the surface at the point P.

    (b) Find the equation of the normal line to the surface at the point P.

39. A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate (including the boundary $x^2 + y^2 = 1$) is heated so that the temperature at any point (x, y) on the plate is given by

$$T(x, y) = x^2 + 2y^2 - x$$

Find the temperatures at the hottest and the coldest points on the plate, including the boundary $x^2 + y^2 = 1$.

40. The acceleration of a particle at any time t is given by

$$\mathbf{a}(t) = \langle -3\cos t, -3\sin t, 2\rangle,$$

while its initial velocity is $v(0) = \langle 0, 3, 0\rangle$. At what times, if any are the velocity and the acceleration of the particle orthogonal?

Spring 2009 Exam

41. For the following parts, consider the two lines

$$\mathbf{r}_1(t) = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$$
$$\mathbf{r}_2(s) = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle$$

a) Find the point at which the given lines intersect.

b) Find a normal vector of the plane which contains these two lines.

c) Find the cosine of the angle between the lines $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$.

42. (a) Consider the surface **S** defined by the equation $xy^2 + xz + z^2 = 7$. Find the equation of the tangent plane to the surface **S** at the point (1,1,2).

(b) Find the directional derivative of $f(x, y, z) = x^2 + xy^2 + z$ at the point $(1, 2, 3)$ in the direction $\mathbf{v} = \langle 1, 2, 2\rangle$.

43. (a) Find the volume of the solid under the graph of $z = 9 - x^2 - y^2$, inside the cylinder $x^2 + y^2 = 1$, and above the xy-plane.

(b) Evaluate the iterated integral

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx.$$

44. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f satisfying $\nabla f = \mathbf{F}$.

    (a) $\mathbf{F}(x, y) = \langle 3e^{2y} + 2xy, xe^{2y} + x^2 + y^2 \rangle$.

    (b) $\mathbf{F}(x, y) = (3x^2 + 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$.

45. (a) Find parametric equations for the line segment C from the point $(-1, 5, 0)$ to the point $(1, 6, 4)$.

    (b) Evaluate the line integral

$$\int_C xz^2\, dy + y\, dz,$$

    where C is the line segment in part (a).

46. Let D be the region on the plane bounded by the curves $y = 2x - x^2$ and the x-axis, and C be the positively oriented boundary curve of D. Use **Green's Theorem** to evaluate the line integral

$$\oint_C (xy + \cos(e^x))\, dx + (x^2 + e^{\cos y})\, dy.$$

More Problems

47. Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

48. Find the equation of the plane containing the points $P(1, 3, 0)$, $Q(2, -1, 2)$ and $R(0, 0, 1)$.

49. Find all points of intersection of the parametric curve $\mathbf{r}(t) = \langle 2t^2 - 2, t, 1 - t - t^2 \rangle$ and the plane $x + y + z = 3$.

50. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + 2y^2 - 2y$ on the closed disc $x^2 + y^2 \leq 5$ of radius $\sqrt{5}$.

51. Evaluate

$$\int\int_R xy\, dA,$$

    where R is the region in the first quadrant bounded by the line $y = 2x$ and the parabola $y = x^2$.

52. Consider the vector field $\mathbf{F}(x, y) = \langle 2xy + \sin y, x^2 + x\cos y + 1 \rangle$.

    (a) Show that $\mathbf{F}(x, y) = \langle 2xy + \sin y, x^2 + x\cos y + 1 \rangle$ is conservative by finding a potential function $f(x, y)$ for $\mathbf{F}(x, y)$.

    (b) Use your answer to part **a** to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the arc of the parabola $y = x^2$ going from $(0, 0)$ to $(2, 4)$.

53. Evaluate the line integral
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$
where $\mathbf{F} = \langle y^2 + \sin x, xy \rangle$ and C is the unit circle oriented counterclockwise.

54. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y^2, 2xy + x \rangle$ and C is the curve starting at $(0, 0)$, traveling along a line segment to $(2, 1)$ and then traveling along a second line segment to $(0, 3)$.

Fall 2009 Exam

55. Let $f(x, y) = 3x^2/2 + 3y/2 + 2$.
    (a) Compute the gradient of f.
    (b) Find an equation for the tangent plane P to the graph of f at the point $(1, 1, 5)$.
    (c) Compute the distance from the plane P to the origin.

56. Let $f(x, y) = 13x^2 + 16xy - 8x + 5y^2 - 6y + 5$. Find and classify the critical points of f.

57. Find the volume of the solid that lies below the graph of $f(x, y) = 1 + x^2$ and above the region in the first quadrant between the graphs of $y = x$ and $y = x^3$.

58. The figure shows the graph of $r = \sin(3\theta)$, the three-leaved rose. Find the area enclosed by one loop of the graph.

59. Let $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + (2t^3/3 - 3t - 1)\mathbf{k}$ be the position function of a particle in motion.
    (a) Determine the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ of the particle.
    (b) Find all times t such that the velocity is perpendicular to the acceleration.

60. Let C be the path from $(0, 0)$ to $(1, 1)$ along the graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$. (Note that the path C is the same in (a) and (b).)
    (a) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (3x^2/2 + y^2)\mathbf{i} + (2x^2 + y)\mathbf{j}$
    (b) Evaluate $\int_C \mathbf{G} \cdot d\mathbf{r}$, where $\mathbf{G} = (3x^2y + 2xy + 1)\mathbf{i} + (x^3 + x^2)\mathbf{j}$.

61. Let C be the closed path of four straight line segments obtained by starting at $(1, 0)$ and going successively to $(0, -1), (-1, 0), (0, 1)$, then back to $(1, 0)$. Compute
$$\oint_C (\sin(x) + y^3)\, dx + (3xy^2 + x + e^y)\, dy.$$