HURWITZ EQUIVALENCE FOR
LEFSCHETZ FIBRATIONS AND THEIR MULTISESSIONS

R. İNANÇ BAYKUR AND KENTA HAYANO

Abstract. In this article, we characterize isomorphism classes of Lefschetz fibrations with multisections via their monodromy factorizations. We prove that two Lefschetz fibrations with multisections are isomorphic if and only if their monodromy factorizations in the relevant mapping class groups are related to each other by a finite collection of modifications, which extend the well-known Hurwitz equivalence. This in particular characterizes isomorphism classes of Lefschetz pencils. We then show that, from simple relations in the mapping class groups, one can derive new (and old) examples of Lefschetz fibrations which cannot be written as fiber sums of blown-up pencils.

1. Introduction

Lefschetz fibrations became a central tool in differential geometry and topology following Donaldson’s insight in the late 1990s [5] that one can effectively explore the topological aspects of manifolds by studying smooth maps on them which have locally holomorphic character. Since then they have gained a prominent role in symplectic topology, where a striking balance between flexibility and rigidity makes it possible to carry out topological constructions and geometric obstructions at the same time. In dimension 4, a firmer grasp on the quickly developing theory of Lefschetz fibrations is pursued via factorizations in mapping class groups of surfaces [12, 13, 11].

Our goal is to better understand how surfaces in symplectic 4–manifolds arise in this setting. Surfaces in 4–manifolds play a crucial role in our exploration of the topology of smooth and symplectic 4–manifolds: they help determining the homeomorphism type, and distinguishing the diffeomorphism type. As observed by Donaldson and Smith [6], any symplectic surface can be seen as a multisection or an n–section of a Lefschetz fibration, which is a branched surface intersecting all the fibers positively n times. In [4], we initiated an extensive study of symplectic surfaces via factorizations in more elaborate mapping class groups of surfaces. The current article aims to add to this effort by refining and exploring the correspondence between symplectic surfaces in symplectic 4–manifolds, multisections of Lefschetz fibrations, and positive factorizations in surface mapping class groups.

An isomorphism between two Lefschetz fibrations \((X_i, f_i)\) with \(n\)–sections \(S_i\), \(i = 1, 2\), is given by a pair of orientation-preserving diffeomorphisms between the total spaces \(X_i\) and the base 2–spheres, which commute with \(f_i\) and match \(S_i\). The mapping class group \(\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})\), which consists of orientation-preserving self-diffeomorphisms of the surface \(\Sigma_g\) preserving the set of marked...
points \( \{s_1, \ldots, s_n\} \), is the host to the lift of any monodromy factorization of \((X_i; f_i)\) prescribed by an \(n\)-section \(S_i\). There are simple modifications of monodromy factorizations that naturally arise from a few choices made in the process of extracting these factorizations from a given fibration—which we describe and study in detail in the later sections. Two monodromy factorizations in \(\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})\) will be called Hurwitz equivalent if one can be obtained from the other by applying a sequence of this finite set of simple modifications.

Our main theorem in this article is the following:

**Theorem 1.1.** For \(g, n \geq 1\) there exists a one-to-one correspondence

\[
\begin{align*}
\text{Genus–}g \text{ Lefschetz fibrations} & \quad \leftrightarrow \quad \text{Monodromy factorizations} \\
\text{with } n \text{–sections,} & \quad \text{in } \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\}), \\
\text{up to isomorphism} & \quad \text{up to Hurwitz equivalence}
\end{align*}
\]

When \(g \geq 2\) and \(n = 0\), i.e. when the multisections are omitted, the above result is classical, due to Kas [12] and Matsumoto [13].

Theorem 1.1 is proved in Section 3, after a review of background results in Section 2. We also provide an extension of this correspondence to one between framed multisections of Lefschetz fibrations and monodromy factorizations in framed mapping class groups introduced in [4]. In turn, through their monodromy factorizations, we obtain a full characterization of Lefschetz pencils, up to isomorphisms that can permute base points; see Corollary 3.9.

In the last section, we turn to an intriguing question regarding the diversity of Lefschetz fibrations versus that of pencils, which allows us to demonstrate how geometric, topological, and algebraic aspects of the theory of Lefschetz fibrations can be nicely brought together. In [16], Stipsicz asked whether every Lefschetz fibration can be obtained as fiber sums of blown-up pencils; that is to say, whether Lefschetz pencils are the building blocks of all Lefschetz fibrations via blow-ups and fiber sums. In Section 4, we will illustrate a way to produce counter-examples, using the well-known 5–chain relation in the mapping class group of a genus–2 surface, along with monodromy modifications involving multisections, and a variety of geometric and topological results packaged in a recipe from [4] we will be following here. We moreover show –by appealing to above Hurwitz equivalences– that the only other counter-example we know of, a genus–2 Lefschetz fibration of Auroux (shown to be a counter-example by Sato in [14]), can also be derived from the same scheme.

**Acknowledgements.** The results of this article were partially presented by the second author at the 13th International Workshop on Real and Complex Singularities held in August 2014 at Sao Carlos, Brazil. He would like to thank the organizers of the workshop for the opportunity and for their invitation to prepare this manuscript. The first author was partially supported by the Simons Foundation Grant 317732. The second author was supported by JSPS and CAPES under the Japan-Brazil research cooperative program and JSPS KAKENHI (26800027).
2. Multisections of Lefschetz fibrations via positive factorizations

In this section we will briefly review the basic definitions and properties of Lefschetz fibrations and their multisections, focusing on how they can be captured and studied as certain factorizations in mapping class groups of surfaces. For a more detailed exposition, the reader can turn to [11] and [4].

Throughout the paper, all manifolds we work with are assumed to be closed, connected and oriented, unless otherwise noted.

2.1. Lefschetz pencils, fibrations, and multisections.

A Lefschetz fibration \((X, f)\) is a smooth map \(f : X \to S^2\), from a 4–manifold \(X\) onto the 2–sphere, which only has nodal singularities, that is, for any \(x\) in the critical locus \(\text{Crit}(f)\), there exist orientation-preserving complex coordinate neighborhoods \((U, \varphi)\) at \(x \in X\) and \((V, \psi)\) at \(f(x) \in S^2 \cong \mathbb{CP}^1\), such that \(\psi \circ f \circ \varphi^{-1}(z, w) = z^2 + w^2\).

So \(f\) is a submersion at all but finitely many points, where we have the local model of a complex nodal singularity. For \(g\) the genus of a regular fiber, we call \((X, f)\) a genus–\(g\) Lefschetz fibration.

A Lefschetz pencil \((X, f)\) is a Lefschetz fibration \(f : X \setminus B \to S^2\), where \(B\) is a non-empty discrete set in \(X\), called the base locus, such that there exist an orientation-preserving complex coordinate neighborhood \((U, \varphi)\) around each base point \(x \in B\) and a diffeomorphism \(\psi : S^2 \to \mathbb{CP}^1\), which together satisfy \(\psi \circ f \circ \varphi^{-1}(z, w) = [z : w]\).

We say \((X, f)\) is a genus–\(g\) Lefschetz pencil with \(n\) base points for \(g\) the genus of the regular fiber (compactified by adding the base points), and \(n = |B|\). Given a genus–\(g\) Lefschetz pencil \((X, f)\) with \(n\) base points, we can obtain a genus–\(g\) Lefschetz fibration \(f' : X' = X \# n\mathbb{CP}^2 \to S^2\) with \(n\) disjoint sections \(S_j\) of self-intersection \(-1\), each arising as an exceptional sphere of the blow-up at the base point \(x_j\). The correspondence is canonical, as one can blow-down all the \(S_j\) in the fibration \((X', f')\) to recover the pencil \((X, f)\) (e.g. [11 §3.1]).

Recall that a symplectic structure is a closed non-degenerate 2–form \(\omega\) on a smooth manifold, such as the Kähler form on a complex algebraic variety. A symplectic 4–manifold is then a pair \((X, \omega)\). The prominent role of Lefschetz fibrations in differential geometry and topology is mostly due to Donaldson’s amazing result from the late 1990s, who showed that an analogue of the classical Lefschetz hyperplane theorem for complex algebraic surfaces holds in this more flexible setting: every symplectic 4–manifold admits a Lefschetz pencil [5]. Conversely, generalizing an idea of Thurston, Gompf observed that every 4–manifold admitting a Lefschetz pencil or a non-trivial (i.e. with non-empty critical locus) Lefschetz fibration is symplectic [11]. Furthermore, one can strike a compatibility condition between the pairs \((X, \omega)\) and \((X, f)\), which asks for the fibers of \(f\) to be symplectic surfaces with respect to \(\omega\).

**Theorem 2.1** (Donaldson, Gompf). Every symplectic 4–manifold \((X, \omega)\) admits a compatible Lefschetz pencil, and every Lefschetz pencil / non-trivial fibration \((X, f)\) can be equipped with a compatible symplectic form.
The main companion of a Lefschetz fibration \((X, f)\) in this paper will be an embedded surface \(S\) which sits in \(X\) in a rather special way with respect to \(f\).

**Definition 2.2** (§). A (possibly disconnected) closed oriented surface \(S \subset X\) is called a **multisection**, or an \(n\)–**section**, of a Lefschetz fibration \((X, f)\) if it satisfies the conditions:

1. The restriction \(f|_S\) is an \(n\)–fold simple branched covering,
2. The restriction of the differential \(df_x : N_x S \to T_{f(x)} S^2\) preserves the orientation for any branched point \(x \in S\) of \(f|_S\), where \(N_x S \subset T_x X\) is a normal space of \(T_x S\), which has the canonical orientation derived from that of \(X\),
3. For any branched point \(x \in S \cap \text{Crit}(f)\) of \(f|_S\), there exist complex coordinate neighborhoods \((U, \varphi)\) and \((V, \psi)\) which make the following diagram commute:

\[
\begin{array}{ccc}
(U, U \cap S) & \xrightarrow{\varphi} & (\mathbb{C}^2, \mathbb{C} \times \{0\}) \\
\downarrow f & & \downarrow (z, w) \mapsto z^2 + w^2 \\
V & \xrightarrow{\psi} & \mathbb{C}.
\end{array}
\]

A triple \((X, f, S)\) will denote a Lefschetz fibration \((X, f)\) and its multisection \(S\).

Just like how a Lefschetz fibration locally behaves like a holomorphic map, a multisection behaves like a holomorphic curve, intersecting the fibers all positively, and so that the restriction of the fibration map to it is a holomorphic branched covering onto \(\mathbb{CP}^1\). Multisections are found in abundance, as observed by Donaldson and Smith (who referred to them as standard surfaces): for any symplectic surface \(S\) in a symplectic 4–manifold \((X, \omega)\), there exists a compatible Lefschetz pencil \((X, f)\), such that \(S\) is a multisection of \(f|_{X \setminus B}\), and conversely, for any triple \((X, f, S)\), there exists a compatible symplectic form \(\omega\) making the fibers and \(S\) symplectic [6].

2.2. **Local and global monodromies, positive factorizations.**

We will make a few additional assumptions on \((X, f, S)\), merely to simplify our upcoming discussions on how to describe Lefschetz fibrations and their multisections in terms of certain factorizations in surface mapping class groups. First, we will assume that \(f\) is injective on \(\text{Crit}(f)\), and also that each branched point of a multisection, if not contained in \(\text{Crit}(f)\), does not lie on a singular fiber, i.e. not contained in \(f^{-1}(\text{Crit}(f))\). Both of these can be always achieved after a small perturbation. These assumptions will allow us to get standard local models for \((X, f, S)\) for the fibration over \(S^2 \setminus f(\text{Crit}(f) \cup \text{Crit}(f|_S))\) –and not only around the isolated points in \(\text{Crit}(f) \cup \text{Crit}(f|_S)\).

It is also customary to assume that \(f\) is relatively minimal, that is, no fiber contains a sphere with self-intersection \(-1\), which otherwise could be blown-down without destroying the rest of the fibration. A Lefschetz pencil \((X, f)\) is said to be relatively minimal, if no fiber component is a self-intersection \(-1 + k\) sphere containing \(k\) points of \(B\), to ensure that the associated Lefschetz fibration \((X', f')\) is relatively minimal. As we will see shortly, this assumption is needed to guarantee that no information on the local topology is lost when we look at the monodromy of the fibration.
For a surface $\Sigma$ and points $s_1, \ldots, s_n \in \Sigma$, let $\text{Diff}(\Sigma; \{s_1, \ldots, s_n\})$ be the group of orientation-preserving diffeomorphisms of $\Sigma$ which preserve the set $\{s_1, \ldots, s_n\}$. We call $\text{Mod}(\Sigma; \{s_1, \ldots, s_n\}) = \pi_0(\text{Diff}(\Sigma; \{s_1, \ldots, s_n\}))$ the mapping class group of $\Sigma$ with marked points $\{s_1, \ldots, s_n\}$. It consists of elements of $\text{Diff}(\Sigma; \{s_1, \ldots, s_n\})$, modulo isotopies fixing the set $\{s_1, \ldots, s_n\}$, where the group structure is induced by compositions of maps, that is, $[\varphi_1] \cdot [\varphi_2] = [\varphi_1 \circ \varphi_2]$ for $\varphi_1, \varphi_2 \in \text{Diff}(\Sigma; \{s_1, \ldots, s_n\})$.

Let $\Sigma_n$ denote a genus-$g$ surface with $n$ boundary components, and take points $u_1, \ldots, u_n \in \partial \Sigma_n$ which cover the elements $\pi_0(\Sigma_n^g)$. The framed mapping class group $\text{Mod}(\Sigma_n^g; \{u_1, \ldots, u_n\})$ consists of orientation-preserving diffeomorphisms of $\Sigma_n^g$ which preserve set of marked points $\{u_1, \ldots, u_n\}$, modulo isotopies fixing the same data $\mathcal{H}$. Clearly, one can pass to a closed surface $\Sigma_g = \Sigma_n^g$ by capping the boundaries by disks, the centers of which we label as $s_1, \ldots, s_n \in \Sigma_g$. This boundary capping map induces a surjective homomorphism

$$\text{Mod}(\Sigma_n^g; \{u_1, \ldots, u_n\}) \twoheadrightarrow \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\}).$$

Now let $(X, f)$ be a Lefschetz fibration with $l$ critical points, $S$ its $n$–section and $\text{Crit}(f|_S) = \{b_1, \ldots, b_k\} \subset X$ the set of branched points of $f|_S$ away from $\text{Crit}(f)$. Let $f|_{\text{Crit}(f) \cup \text{Crit}(f|_S)} = \{a_1, \ldots, a_{k+l}\}$, and take paths $\alpha_1, \ldots, \alpha_{k+l} \subset S^2$ with a common initial point $p_0 \in S^2 \setminus f(\text{Crit}(f) \cup \text{Crit}(f|_S))$ such that

- $\alpha_1, \ldots, \alpha_{k+l}$ are mutually disjoint except at $p_0$,
- $\alpha_i$ connects $p_0$ with $a_i$,
- $\alpha_1, \ldots, \alpha_{k+l}$ are ordered counterclockwise around $p_0$, i.e. there exists a small loop around $p_0$ oriented counterclockwise, hitting each $\alpha_i$ only once in the given order.

We take a loop $\tilde{\alpha}_i$ with the base point $p_0$ by connecting $\alpha_i$ with a small counterclockwise circle with center $a_i$. Let $\mathcal{H}$ be a horizontal distribution of $f|_{X \setminus (\text{Crit}(f) \cup \text{Crit}(f|_S))}$, that is, $\mathcal{H} = \{\mathcal{H}_x\}_{x \in X \setminus (\text{Crit}(f) \cup \text{Crit}(f|_S))}$ is a plane field such that $\text{Ker}(d\tilde{\alpha}_x) \oplus \mathcal{H}_x = T_xX$ for any $x \in X \setminus (\text{Crit}(f) \cup \text{Crit}(f|_S))$. We assume that $\mathcal{H}_x = T_xS$ for any $x \in S \setminus \text{Crit}(f|_S)$. Using $\mathcal{H}$, we can take a lift of the direction vector field of $\tilde{\alpha}_i$ and a flow of this lift gives rise to a self-diffeomorphism of $f^{-1}(p_0)$. We call this diffeomorphism a parallel transport of $\tilde{\alpha}_i$ and its isotopy class a local monodromy around $a_i$. Note that a local monodromy does not depend on the choice of $\mathcal{H}$. Indeed, for any horizontal distribution $\mathcal{H}$ we can find a Riemannian metric $g$ such that $\mathcal{H}_x$ is equal to $(\text{Ker}(d\tilde{\alpha}_x))^\perp$, in particular for any two horizontal distribution there exists a one-parameter family of horizontal distributions connecting the given two distributions.

Under an identification of the pair $(f^{-1}(p_0), f^{-1}(p_0) \cap S)$ with $(\Sigma_g, \{s_1, \ldots, s_n\})$, we can regard a parallel transport as a diffeomorphism in $\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\})$, and thus, a local monodromy as a mapping class in $\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$. We denote this mapping class by $\xi_i$. Since the concatenation $\tilde{\alpha}_1 \cdots \tilde{\alpha}_{k+l}$ is null-homotopic in $S^2 \setminus f(\text{Crit}(f) \cup \text{Crit}(f|_S))$, the composition $\xi_{k+l} \cdots \xi_1$ is the unit element of the group $\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$. The factorization

$$\xi_{k+l} \cdots \xi_1 = 1 \quad \text{in } \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$$

is called a monodromy factorization of the triple $(X, f, S)$, which we will denote in short by $W_{X,f,S}$ (as a word in $\xi_i \in \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$).
Analyzing the local models around Crit(\(f\)) and Crit(\(f|_{S}\)) (which, remember, might contain in common the type (3) branched points in Definition 2.2), we identify three standard elements in the mapping class group \(Mod(\Sigma_g; \{s_1, \ldots, s_n\})\) that appear as a factor \(\xi_i\) above [4]: If the fiber \(f^{-1}(a_i)\) contains a Lefschetz critical point which is not a branched point of \(f|_{S}\), then \(\xi_i\) is a right-handed Dehn twist along some simple closed curve \(c \subset \Sigma_g \setminus \{s_1, \ldots, s_n\}\), which is called a vanishing cycle of a Lefschetz critical point in \(f^{-1}(a_i)\) [12]. (Relative minimality of \((X, f)\) now guarantees that \(c\) is not null-homotopic, so we do not have a “hidden” Dehn twist factor.) If \(f^{-1}(a_i)\) contains a branched point of \(S\) away from Crit(\(f\)), then \(\xi_i\) is a half twist \(\tau_{\gamma_i}\) along some path \(\gamma_i \subset \Sigma_g\) between some \(s_j\) and \(s_j'\). Lastly, if \(a_i\) is the image of a point in Crit(\(f\)) \(\cap\) Crit(\(f|_{S}\)), we get a mapping class \(\xi_i\) which is represented by a Dehn twist \(t_c\) shown in Figure 1 for some simple closed curve \(c \subset \Sigma_g\) going through \(s_j\) and \(s_j'\).

Observe that under the forgetful homomorphism, only \(\xi_i\) that are Dehn twists (possibly going through Crit(\(f|_{S}\))) survive, yielding the standard monodromy factorization \(W_{X,f}\) of \((X, f)\) of the form \(t_{\alpha_1} \cdots t_{\alpha_l} = 1\) in \(Mod(\Sigma_g)\). In other words, the factorization \(W_{X,f,S}\) is a lift of the factorization \(W_{X,f}\).

It is worth noting that each standard element \(\xi_i\) discussed above, let it be a Dehn twist or an arc twist, comes with a preferred orientation, corresponding to positive (right-handed) Dehn twists and arc/braid twists. Any factorization \(\xi_{k+1} \cdots \xi_1 = 1\) in \(Mod(\Sigma_g; \{s_1, \ldots, s_n\})\), which consists of only these three types of elements is called a positive factorization (of the identity) in \(Mod(\Sigma_g; \{s_1, \ldots, s_n\})\), and it conversely gives rise to a triple \((X, f, S)\).

We can summarize these as follows (which is a direct corollary of [4, Theorem 1.1] obtained by applying the boundary capping homomorphism to the framed mapping class group):

**Theorem 2.3.** [4] Let \((X, f, S)\) be a genus-\(g\) Lefschetz fibration with \(l\) critical points, where \(S\) is a connected \(n\)-section which has \(k\) branched points away from Crit(\(f\)) and \(r\) branched points on Crit(\(f\)). Then \((X, f, S)\) has a monodromy factorization \(W_{X,f,S}\) of the form \(\xi_{k+1} \cdots \xi_1 = 1\) in \(Mod(\Sigma_g; \{s_1, \ldots, s_n\})\), where among \(\xi_i\), \(k\) many are half-twists \(\tau_{\gamma_i}\), \(r\) many are Dehn twists \(t_{c_i}\) through two marked points in \(\{s_1, \ldots, s_n\}\), and the rest are Dehn twists along curves missing the marked points. Moreover, all \(s_j\) are hit by the collection of \(\alpha_i\) and \(c_i\), and \(g(S) = \frac{1}{2}(k+r) - n + 1\).

Conversely, from any such positive factorization of 1 in \(Mod(\Sigma_g; \{s_1, \ldots, s_n\})\), subject to conditions listed above, one can construct a genus-\(g\) Lefschetz fibration \((X, f)\) with \(l\) vanishing cycles \(c_1, \ldots, c_l\), and a connected \(n\)-section \(S\) of genus \(g(S) = \frac{1}{2}(k+r) - n + 1\).
Remark 2.4. As we have shown in [4] the positive factorization $W_{X,f,S}$ of the identity element in $\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$ lifts to another positive factorization of a product of boundary parallel Dehn twists in $\text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\})$. This geometrically corresponds to removing a framed tubular neighborhood of $S$. The latter positive factorization consists of standard factors $t_{c_i}$, which are Dehn twists in the interior of $\Sigma^n_g$, and the lifts of $\tau_\gamma$ and $\tilde{t}_c$ as shown in Figure 2, both of which interchange the two boundary components in prescribed ways.

For simplicity, we denote these lifts by $\tau_\gamma$ and $\tilde{t}_c$ as well. This monodromy factorization in the framed mapping class group allows us to capture the self-intersection number of $S$.

3. Equivalence of Lefschetz fibrations with multisections

The goal of this section is to establish a one-to-one correspondence between triples $(X, f, S)$, where $S$ is an $n$–section of a genus–$g$ Lefschetz fibration $(X, f)$, and positive factorizations $W_{X,f,S}$ of the form $\xi_{k+l} \cdots \xi_1 = 1$ in $\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$, modulo natural equivalence relations on both sides, which we will spell out right away.

The triples $(X_i, f_i, S_i)$, $i = 1, 2$, are said to be equivalent (or isomorphic) if there exist orientation-preserving diffeomorphisms $\Phi : X_1 \to X_2$ and $\phi : S^2 \to S^2$ such that $\Phi(S_1) = S_2$ and $f_2 \circ \Phi = \phi \circ f_1$. Clearly, a necessary condition for $(X_i, f_i, S_i)$ to be equivalent is that both fibrations $f_i$ should have the same genus $g$, and the multisections $S_i$ should have the same covering degree $n$.

As we noted in the previous section, a monodromy factorization $(X, f, S)$ does not depend on the choice of a horizontal distribution $\mathcal{H}$. It does however depend on the choice of paths $\alpha_1, \ldots, \alpha_{k+l}$ and that of an identification of $(f^{-1}(p_0), f^{-1}(p_0) \cap S)$ with $(\Sigma_g, \{s_1, \ldots, s_n\})$. Identical to the well-known case of a monodromy factorization of a pair $(X, f)$ one can (see e.g. [11]) easily verify that two monodromy factorizations for a triple $(X, f, S)$, derived from different choices of paths and identifications can be related by successive applications of the following two types of modifications:

(1) **Elementary transformation**, which changes a factorization as follows:

$$\xi_{k+l} \cdots \xi_{i+1} \xi_i \cdots \xi_1 \leftrightarrow \xi_{k+l} \cdots \xi_{i+1} (\xi_i^{-1} \xi_{i+1} \xi_i \cdots \xi_1).$$
Global conjugation, which changes each member of a factorization by the conjugation of some mapping class \( \psi \in \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\}) \):
\[
\xi_k \xi_{k+1} \cdots \xi_1 \longleftrightarrow (\psi^{-1} \xi_k \psi) (\psi^{-1} \xi_{k+1} \psi) \cdots (\psi^{-1} \xi_1 \psi).
\]
We will thus call two factorizations of the unit element \( 1 \in \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\}) \) Hurwitz equivalent if one can be obtained from the other after a sequence of these two types of modifications.

3.1. Equivalence of monodromy factorizations.

This subsection will be devoted to the proof of the following theorem, which, together with Theorem 2.3, implies the main result of our paper, Theorem 1.1.

**Theorem 3.1.** Let \((X_i, f_i, S_i)\), \(i = 1, 2\) be a genus–\(g\) Lefschetz fibration with an \(n\)–section \(S_i\). Suppose that \(2 - 2g - n\) is negative, that is, \(f_i^{-1}(p_0) \setminus (f_i^{-1}(p_0) \cap S_i)\) is a hyperbolic surface for a regular value \(p_0\). The triples \((X_1, f_1, S_1)\) and \((X_2, f_2, S_2)\) are equivalent if and only if their monodromy factorizations \(W_{X_1, f_1, S_1}\) and \(W_{X_2, f_2, S_2}\) are Hurwitz equivalent.

To prove Theorem 3.1, we will need a few preliminary results on mapping classes in \(\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})\).

**Lemma 3.2.** Let \(\gamma_1, \gamma_2 \subset \Sigma_g\) be simple paths between distinct marked points \(s_i\) and \(s_j\). Then \(\tau_{\gamma_1} = \tau_{\gamma_2}\) in \(\text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})\) if and only if \(\gamma_1\) and \(\gamma_2\) are isotopic relative to the points \(s_1, \ldots, s_n\).

**Proof.** The “if part” is obvious. To prove the “only if” part we assume that \(\gamma_1\) and \(\gamma_2\) are not isotopic and show that \(\tau_{\gamma_1}\) and \(\tau_{\gamma_2}\) are not equal. For simple curves \(d_1, d_2\), we denote the geometric intersection number of \(d_1\) and \(d_2\) by \(i(d_1, d_2)\), that is, \(i(d_1, d_2)\) is the minimum number of intersections between two curves isotopic (relative to \(s_1, \ldots, s_n\)) to \(d_1\) and \(d_2\).

Without loss of generality we may assume that \(\gamma_1\) and \(\gamma_2\) are in minimal position. Let \(d\) be the boundary of a regular neighborhood of \(\gamma_1\). The curve \(d\) does not intersect \(\gamma_1\). On the other hand, \(i(d, \gamma_2)\) is not equal to 0. To see this, we will check that there is no bigon between sub-paths of \(d\) and \(\gamma_2\) (see the bigon criterion in [10]). As shown in Figure 3, there are three types of regions which are candidates of such bigons. The two shaded regions in Figure 3(a) cannot be bigons since \(\gamma_1\) and

![Figure 3](image-url)

**Figure 3.** Candidates of bigons around \(d\).
\( \gamma_2 \) are in minimal position. Since \( \gamma_1 \) and \( \gamma_2 \) are not isotopic, the shaded region in Figure 3(b) is not a bigon.

Since \( i(d, \tau_{\gamma_1}(d)) \) is equal to 0, it is sufficient to prove that \( i(d, \tau_{\gamma_2}(d)) \) is not equal to 0. Let \( d' \) be the simple closed curve obtained by changing a parallel copy of \( d \) around \( \gamma_2 \) as shown in Figure 4. The curve \( d' \) represents the isotopy class \( \tau_{\gamma_2}(d) \). It is easy to see that the number of the intersections between \( d \) and \( d' \) is equal to 2\( i(d, \gamma_2) \). In what follows, we will prove that \( d \) and \( d' \) are in minimal position using the bigon criterion. As shown in Figure 5, there are six types of regions which are candidates of innermost bigons. If the far right region in Figure 5(a) were a bigon, \( d \) would be isotopic to a small circle with center \( s_j \), but it is not the case since \( d \) does not intersect \( \gamma_1 \). Similarly, we can also verify that the far left region in Figure 5(b) is not a bigon. If either of the rest of two regions in Figure 5(a) were a bigon, then \( d \) and \( \gamma_2 \) would form a bigon, which contradicts the assumption that \( d \) and \( \gamma_2 \) are in minimal position. As for the rest of two regions in Figure 5(b), either the boundary of them contain at least two sub-paths of \( d \), or they contain the point \( s_i \) or \( s_j \). In either case, these regions cannot be bigons. We can eventually conclude that \( d' \) and \( d \) are in minimal position, and thus, \( i(d, \tau_{\gamma_2}(d)) = 2i(d, \gamma_2)^2 \neq 0. \)

**Lemma 3.3.** Let \( c_1, c_2 \subset \Sigma_g \) be simple closed curves going through \( s_i \) and \( s_j \). Then \( \tilde{\tau}_{c_1} = \tilde{\tau}_{c_2} \) in \( \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\}) \) if and only if \( c_1 \) and \( c_2 \) are isotopic relative to the points \( s_1, \ldots, s_n \).

**Proof.** Once again the “if part” of the statement is obvious. Assume that \( c_1 \) and \( c_2 \) are not isotopic. We will prove that \( \tilde{\tau}_{c_1} \) and \( \tilde{\tau}_{c_2} \) are not equal.

By an isotopy relative to the points \( s_1, \ldots, s_n \), we change \( c_1 \) and \( c_2 \) so that these are in minimal position. Furthermore, we may assume that, at each of the points...
s_i and s_j, either c_1 and c_2 intersect transversely or these are tangent to each other. We take a parallel copy c'_1 of c_1 as shown in Figures 6(a), 6(b) and 6(c). We can easily verify that no region in Figures 6(a), 6(b) and 6(c) can be a bigon. Thus c'_1 and c_2 are in minimal position if c_1 and c_2 intersect at more than two points. Suppose that c_1 and c_2 intersect only at s_i and s_j. If one of the two shaded regions in Figure 6(d) is a bigon, then we can move c_2 by a homotopy so that c_1 and c_2 are tangent to each other at s_i and s_j. After applying this isotopy, we retake a simple closed curve c'_1 as shown in Figure 6(f). The curves c'_1 and c_2 in Figure 6(e) are in minimal position since these intersect on one point transversely. Either of the shaded regions in Figure 6(f) is not a bigon since c_1 is not isotopic to c_2. We can eventually conclude that the geometric intersection number i(c'_1, c_2) is not equal to 0.

Since c'_1 is disjoint from c_1, \(i(\tilde{\tau}_{c_1}(c'_1), c'_1)\) is equal to 0. On the other hand, we can easily verify \(i(\tilde{\tau}_{c_2}(c'_1), c'_1) \neq 0\) in a way similar to that in the proof of [10, Fact 3.6]. Thus, \(\tilde{\tau}_{c_1}\) and \(\tilde{\tau}_{c_2}\) are different mapping classes. □

**Remark 3.4.** We should point out that one cannot state Lemmas 3.2 and 3.3 for the lifts of the elements featured in them to the framed mapping class group \(\text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\})\) instead. For example, for the paths \(\gamma\) and \(\gamma' = t_{\delta_i} t_{\delta_j} (\gamma)\) between s_i and s_j, which are not isotopic (relative to the boundary of \(\Sigma^n_g\)) in general, we can see that \(\tau_{\gamma}\) is equal to \(\tau_{\gamma'}\) in \(\text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\})\). There are in

![Figure 6. The curve c'_1 and candidates of bigons.](image-url)
Next is a variation of a classical result of Earle and Schatz [8]:

**Lemma 3.5.** If $2 - 2g - n$ is negative, then $\pi_1(\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\}), \text{id})$ is trivial.

**Proof.** As in the proof of [10, Theorem 4.6], we can obtain the following exact sequence (note that we omit the base points for simplicity):

$$
\pi_2(F_{0,n}(\Sigma_g)) \rightarrow \pi_1(\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\})) \rightarrow \pi_1(\text{Diff}(\Sigma_g)) \\
\rightarrow \pi_1(F_{0,n}(\Sigma_g)),
$$

(1)

where $F_{0,n}(\Sigma_g)$ is the configuration space defined in [9], which is aspherical if $g \geq 1$. Since the group $\pi_1(\text{Diff}(\Sigma_g))$ is trivial for $g \geq 2$ (2), so is $\pi_1(\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\}))$ if $g \geq 2$. Furthermore, the following diagram commutes:

$$
\begin{array}{ccc}
\pi_1(\text{Diff}(T^2)) & \xrightarrow{ev} & \pi_1(F_{0,n}(T^2)) \\
\pi_1(T^2) \downarrow & & \downarrow \\
\end{array}
$$

where $\pi_*$ is induced by the natural projection and $ev_*$ is induced by the evaluation map, which is the same map as that in 1. In particular, $ev_*$ is injective. We can thus deduce from the exact sequence [1] that $\pi_1(\text{Diff}(T^2; \{s_1, \ldots, s_n\}))$ is trivial for $n > 0$.

We can also obtain the following exact sequence:

$$
\pi_2(F_{n-1,1}(S^2)) \rightarrow \pi_1(\text{Diff}(S^2; \{s_1, \ldots, s_{n-1}\})) \rightarrow \pi_1(\text{Diff}(S^2; \{s_1, \ldots, s_{n-1}\})) \\
\rightarrow \pi_1(F_{n-1,1}(S^2)).
$$

The configuration space $F_{n-1,1}(S^2)$ is aspherical for $n \geq 2$ (3). Thus, if the fundamental group $\pi_1(\text{Diff}(\Sigma_g; \{s_1, \ldots, s_{n-1}\}))$ is trivial, so is $\pi_1(\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\}))$. Using (2) we can verify that $\pi_1(\text{Diff}(S^2; \{s_1\}))$ is trivial. We can conclude that $\pi_1(\text{Diff}(S^2; \{s_1, \ldots, s_{n-1}\}))$ is also an infinite cyclic group generated by $\phi_{2\pi\theta}$. It is easy to see that $[\phi_{2\pi\theta}] \in \pi_1(\text{Diff}(S^2; \{s_1, s_2\}))$ is trivial.

**Remark 3.6.** The above lemma can be possibly derived as a corollary of the contractibility of the identity component of $\text{Diff}(\Sigma_g^0)$, as shown in [8], provided $\text{Diff}(\Sigma_g^0)$ is seen to be homotopy equivalent to $\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\})$. For our purposes however, it is sufficient to calculate the fundamental group of $\text{Diff}(\Sigma_g; \{s_1, \ldots, s_n\})$. So we have given a direct proof of Lemma 3.5.

We are now ready to prove the theorem.

**Proof of Theorem 3.1.** We first assume that $(X_1, f_1, S_1)$ and $(X_2, f_2, S_2)$ are equivalent. We can take diffeomorphisms $\Phi : X_1 \rightarrow X_2$ and $\phi : S^2 \rightarrow S^2$ such that $\phi \circ f_1 = f_2 \circ \Phi$ and $\Phi(S_1) = S_2$. Let $\alpha_1, \ldots, \alpha_{k+l} \subset S^2$ be reference paths for $f_1$.
with the common initial point \( p_0 \) which give rise to the factorization \( W_{X_1, f_1, S_1} \) under an identification \( \Theta : (f_1^{-1}(p_0), f_1^{-1}(p_0) \cap S_1) \rightarrow (\Sigma_g, \{s_1, \ldots, s_n\}) \). Let \( p_0' = \phi(p_0) \), \( \alpha_i' = \phi(\alpha_i) \) and \( \Theta' = \Theta \circ \Phi^{-1} \), which is an identification of \( (f_2^{-1}(p_0'), f_2^{-1}(p_0') \cap S_2) \) with \( (\Sigma_g, \{s_1, \ldots, s_n\}) \). It is easily verify that the monodromy factorization of \( f_2 \) obtained from \( \alpha_1', \ldots, \alpha_{k+1}' \) and \( \Theta' \) coincides with \( W_{X_1, f_1, S_1} \). Since any two factorizations of \( (X_2, f_2, S_2) \) are Hurwitz equivalent, so are \( W_{X_1, f_1, S_1} \) and \( W_{X_2, f_2, S_2} \).

In what follows we assume that \( W_{X_1, f_1, S_1} \) and \( W_{X_2, f_2, S_2} \) are Hurwitz equivalent. We first consider the case that \( f_1 \) has no critical points and \( f_1 | S_1 \) has no branched points. In this case, \( f_1 \) can be obtained by pasting two trivial surface bundles over the disk so that the marked points corresponding \( n \)-sections match. By Lemma 3.5 such a pasting map is unique up to isotopy preserving fibration structures. Thus \( f_1 \) and \( f_2 \) are equivalent.

Assume that \( f_1 \) has critical points or \( f_1 | S_1 \) has branched points. We can take reference paths \( \alpha_1, \ldots, \alpha_{k+1} \subset S^2 \) for \( f_1 \) so that the local monodromies associated with \( \alpha_1 \) and \( \alpha_2 \) coincide.

By composing a self-diffeomorphism of \( S^2 \) to \( f_2 \), we may assume that \( f_1(\text{Crit}(f_1) \cup \text{Crit}(f_1 | S_1)) \) and \( f_2(\text{Crit}(f_2) \cup \text{Crit}(f_2 | S_2)) \) coincide. Let \( f_1(\text{Crit}(f_1) \cup \text{Crit}(f_1 | S_1)) = \{a_1, \ldots, a_{k+1}\} \) and \( D_i \subset S^2 \) a sufficiently small disk neighborhood of \( a_i \). Since all the local monodromies of \( f_1 \) and \( f_2 \) coincide, we can take a diffeomorphism \( H : f_1^{-1}(S^2 \setminus \Pi_i D_i) \rightarrow f_2^{-1}(S^2 \setminus \Pi_i D_i) \) sending the intersection \( f_1^{-1}(S^2 \setminus \Pi_i D_i) \cup S_1 \) to \( f_2^{-1}(S^2 \setminus \Pi_i D_i) \cup S_2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
C_{f_1^{-1}(D_i)} & \xrightarrow{H} & C_{f_2^{-1}(D_i)} \\
\downarrow f_1 & & \downarrow f_2 \\
S^2 \setminus \Pi_i D_i.
\end{array}
\]

In what follows we will extend \( H \) to a diffeomorphism with the source containing the preimages \( C_{f_1^{-1}(D_i)}, \ldots, C_{f_1^{-1}(D_{k+1})} \).

If \( f_1^{-1}(a_i) \) contains a Lefschetz critical point which is not a branched point of \( f_1 | S_1 \), then in the same manner as in the proof of [13, Theorem 2.4] we can extend \( H \) to a diffeomorphism with the source containing \( C_{f_1^{-1}(D_i)} \) (in this procedure we need Lemma 3.5 instead of the contractibility of \( \text{Diff}(\Sigma_g) \) used in [13, p.133]).

Assume that \( f_1^{-1}(a_i) \) contains a Lefschetz critical point \( x_i \) which is also a branched point of \( f_1 | S_1 \). There exist complex coordinate neighborhoods \( (U^i, \varphi^i) \) and \( (V^i, \psi^i) \) at \( x_i \in X \) and \( f_1(x_i) \in S^2 \), respectively, which satisfy the following properties:

1. \( \psi^i \circ f_1 \circ (\varphi^i)^{-1}(z, w) = z^2 + w^2 \),
2. \( \varphi^i(U^i \cap S_1) = \mathbb{C} \times \{0\} \).

Using the disk theorem as in the proof of [13, Theorem 2.4], we may assume that \( \psi^i \) and \( \psi^j \) coincide without loss of generality. By Lemma 3.3 the vanishing cycles associated with \( \alpha_1 \) and \( \alpha_2 \) coincide up to isotopy relative to the points \( s_1, \ldots, s_n \).

Thus, in the same way as that in the proof of [13, Lemma 2.5], we can change \( H \) by a vertical isotopy (in the sense of [13]) sending \( S_1 \) to \( S_2 \) at all times so that \( \varphi^2 \circ H = \varphi^1 \) on a neighborhood (in \( f_1^{-1}(\partial D_i) \)) of the vanishing cycle of \( f_1 \) associated
with $\alpha_i^1$. The arguments following the proof of [13 Lemma 2.5] can be applied to our situation, and we can eventually extend $H$ to a diffeomorphism with the source containing $f_i^{-1}(D_i)$.

Lastly, if $f_i^{-1}(a_i)$ contains a branched point of $f_i|_{S_i}$ away from $\text{Crit}(f_i)$, then we can extend $H$ to a diffeomorphism with the source containing $f_i^{-1}(D_i)$ in a manner quite similar to that in the previous paragraph, where we invoke Lemma 3.2 instead of Lemma 3.3 this time.

□

3.2. Monodromy factorizations in the framed mapping class group.

As discussed in the previous section (cf. Remark 2.4) we can take a lift of the factorization $W_{X,f,S}$ to that of a product of Dehn twists along boundary components in the framed mapping class group $\text{Mod}(\Sigma_n^g; \{u_1, \ldots, u_n\})$. Such a lift is needed to fully capture the local topology of the multisection $S$.

Two such lifts $\tilde{W}_{X_i,f_i,S_i}$ of $W_{X_i,f_i,S_i}$, for $i = 1, 2$, are not necessarily related by elementary transformations and simultaneous conjugations even if $(X_1, f_1, S_1)$ and $(X_2, f_2, S_2)$ are equivalent. There is indeed no canonical way to choose lifts of $\tau_\gamma$ and $\tilde{t}_c$. For instance, each one of the paths $\gamma_i$, $i = 1, 2$ in Figure 7 is a lift of $\gamma$, and in turn, $\tau_\gamma_i \in \text{Mod}(\Sigma_n^g; \{u_1, \ldots, u_n\})$ is a lift of $\tau_\gamma \in \text{Mod}(\Sigma_g; \{s_1, \ldots, s_n\})$.

![Figure 7](image)

**Figure 7.** Examples of paths between $s_i$ and $s_j$.

It is easy to see that $\tau_{\gamma_2}$ is equal to $t_{\delta_k} \tau_\gamma t_{\delta_k}^{-1} = \tau_\gamma t_{\delta_k} t_{\delta_k}^{-1}$, where $\delta_k$ is a simple closed curve along the boundary component containing $u_k$. Thus, different choices of lifts of $\tau_\gamma$ and $t_c$ in $W_{X_i,f_i,S_i}$ might yield a factorization $\tilde{W}_{X_i,f_i,S_i}$ with distinct products, in particular the Hurwitz equivalence class of $\tilde{W}_{X_i,f_i,S_i}$ depends on the choices of lifts of members in $W_{X_i,f_i,S_i}$.

For lifts of $\tau_\gamma$ and $t_c$ are uniquely determined up to conjugations by Dehn twists along boundary components however, $\tilde{W}_{X_i,f_i,S_i}$ is uniquely determined up to elementary transformations, simultaneous conjugations, plus a third modification:

(3) **Framing conjugation**, which changes a factorization as follows:

$$\xi_{k+1} \cdots \xi_i \xi_{i-1} \cdots \xi_1 \mapsto \xi_{k+1} \cdots (t_{\delta}^{-1} \xi_i) \xi_{i-1} \cdots \xi_1,$$

where $\delta$ is a simple closed curve along a boundary component of $\Sigma_g^n$.

A framing conjugation does not affect Dehn twists among the $\xi_i$ factors, since any simple closed curve can be isotoped away from the boundary. In particular, this move is not needed to relate monodromy factorizations in $\text{Mod}(\Sigma_n^g; \{u_1, \ldots, u_n\})$ associated to pure multisections, i.e. disjoint union of sections.
We say that two positive factorizations of products of Dehn twists along boundary components in \( \text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\}) \) are \textit{Hurwitz equivalent}, if one can be obtained from the other by a sequence of elementary transformations, simultaneous conjugations and framing conjugations.

For \( i = 1, 2 \), let \( (X_i, f_i, S_i) \) be a genus--\( g \) Lefschetz fibration with an \( n \)--section \( S_i \), with monodromy factorization \( W_{X_i, f_i, S_i} \) in \( \text{Mod}(\Sigma^n_g; \{s_1, \ldots, s_n\}) \). Let \( \tilde{W}_{X, f, S} \) be a lift of \( W_{X_i, f_i, S_i} \), a positive factorization of the form

\[
\tilde{W}_{X, f, S} : \xi_{k+1} \cdots \xi_1 = t_{\delta_1}^{a_1} \cdots t_{\delta_n}^{a_n}
\]

in \( \text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\}) \). Then Theorem 3.1 together with the above observation, thus yields to a one-to-one correspondence in this setting as well:

**Corollary 3.7.** Suppose that \( 2 - 2g - n \) is negative. The triples \( (X_1, f_1, S_1) \) and \( (X_2, f_2, S_2) \) are equivalent if and only if \( \tilde{W}_{X_1, f_1, S_1} \) and \( \tilde{W}_{X_2, f_2, S_2} \) are Hurwitz equivalent.

**Remark 3.8.** It is worth noting that a framing conjugation can also affect the right hand side of a positive factorization

\[
\xi_{k+1} \cdots \xi_1 = t_{\delta_1}^{a_1} \cdots t_{\delta_n}^{a_n}
\]

in the framed mapping class group, simultaneously increasing and decreasing the powers of the involved boundary twists. In order to make the effect of a framing conjugation clear, let us define the following surjective homomorphism

\[
\Lambda : \text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\}) \to \mathfrak{S}_n
\]

defined by the action of a mapping class on the set \( \{u_1, \ldots, u_n\} \), where \( \mathfrak{S}_n \) is the symmetric group of order \( n \). A Dehn twist \( t_c \) is contained in the kernel of \( \Lambda \), while a half twist \( \tau_c \) and a lift \( \tilde{t_c} \) of a Dehn twist are sent to transpositions by \( \Lambda \). Clearly \( \xi t_c \xi^{-1} \) is equal to \( t_{\delta_{\Lambda(\xi)c}} \xi \) for any \( \xi \in \text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\}) \), so a framing conjugation changes the right hand side of the factorization as follows:

\[
\xi_{k+1} \cdots \xi_1 = t_{\delta_1}^{-1} \cdots t_{\delta_n}^{-1}
\]

3.3. **Equivalence of Lefschetz pencils.**

Given a genus--\( g \) Lefschetz pencil \( (X, f) \) with base locus \( B = \{x_1, \ldots, x_n\} \), recall that we can pass to a genus--\( g \) Lefschetz fibration \( (X', f') \), with a distinguished \textit{pure} \( n \)--section \( S \) that consists of \( n \) disjoint sections \( S_j \) of self-intersection \( -1 \), each arising as an exceptional sphere in \( X' = X' \# n\mathbb{CP}^2 \) of the blow-up at the base point \( x_j \in X \). Since one can blow-down all \( S_j \) to recover the pencil \( (X, f) \), we can work with the well-known monodromy factorization \( \tilde{W}_{X', f', S} \) of \( (X', f', S) \) of the form

\[
t_{c_{k+1}} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_n}
\]

in the framed mapping class group \( \text{Mod}(\Sigma^n_g; \{u_1, \ldots, u_n\}) \). Note that this factorization is in fact contained in the subgroup \( \text{Mod}_\partial(\Sigma^n_g) \) (whose elements restrict to identity along \( \partial\Sigma^n_g \)) which only captures \textit{pure} \( n \)--sections, but we need the larger group in order to factor in pencil automorphisms which swap base points, which we will discuss shortly. This associated factorization is what we will call monodromy
factorization of the pencil \((X,f)\), and with the above correspondence in mind, we will denote it simply by \(W_{X,f}\).

Lefschetz pencils \(f_i : X_i \setminus B_i \to S^2, i = 1, 2\), are said to be equivalent if there exist orientation-preserving diffeomorphism \(\Phi : X_1 \to X_2\) and \(\phi : S^2 \to S^2\) such that \(\Phi(B_1) = B_2\) and \(\phi \circ f_1 = f_2 \circ \Phi\). Clearly, for \((X_i,f_i)\) to be equivalent pencils, they should both have the same fiber genus and the same number of base points \(|B_1| = |B_2|\). It now follows from Theorem 3.1 that:

**Corollary 3.9.** Two Lefschetz pencils \((X_i,f_i)\) of genus \(g \geq 1\) with \(n\) base points are equivalent if and only if they have Hurwitz equivalent monodromy factorizations \(W_{X_i,f_i}\) in \(\text{Mod}(\Sigma^n; \{u_1, \ldots, u_n\})\).

**Proof.** Since \(g \geq 1\) and \(n > 0\), we have \(2 - 2g - n < 0\). If \(f_1\) and \(f_2\) are equivalent, then the corresponding pairs of Lefschetz fibrations and these sections are also equivalent. Thus, \(W_{X_1,f_1}\) and \(W_{X_2,f_2}\) are Hurwitz equivalent by Corollary 3.7. Suppose that \(W_{X_1,f_1}\) and \(W_{X_2,f_2}\) are Hurwitz equivalent. Let \(f'_i : X'_i = X_i \# nCP^2 \to S^2\) be the associated Lefschetz fibration. We deduce from Theorem 3.1 and the assumption that there exist diffeomorphisms \(\Phi' : X'_1 \to X'_2\) and \(\phi : S^2 \to S^2\) such that \(\Phi'\) sends the union of exceptional spheres of \(X'_1\) to that of \(X'_2\) and \(\phi \circ f'_1 = f'_2 \circ \Phi'\). So \(\tilde{\Phi}\) induces a diffeomorphism \(\Phi : X_1 \to X_2\) which satisfies \(\Phi(B_1) = B_2\) and \(\phi \circ f_1 = f_2 \circ \Phi\), providing an equivalence between \((X_1,f_1)\) and \((X_2,f_2)\). \(\square\)

**Remark 3.10.** The Hurwitz equivalence in the statement of Corollary 3.9 is not the classical one for fibrations, it is our (extended) Hurwitz equivalence for monodromy factorizations in the framed mapping class group, allowing the exceptional sections / base points to be interchanged. Although we believe the above corollary to be known to experts, we are not aware of any proof of it in the literature.

4. Lefschetz fibrations which do not arise from pencils

Although every Lefschetz pencil gives rise to a Lefschetz fibration on a blow-up of its total space, the converse is known to be false. As shown by Stipsicz [10], and independently by Smith [13], if \((X,f)\) is a fiber sum of two nontrivial Lefschetz fibrations, it cannot have any exceptional sections, i.e. sections of self-intersection \(-1\), and thus it is not a blow-up of a pencil. Motivated by this, Stipsicz conjectured in the same article that every Lefschetz fibration \((X,f)\), if not a blow-up of a pencil, is a fiber sum of such, which amounts to having Lefschetz pencils as building blocks of any Lefschetz fibration via fiber sums.

In [14], Sato proved that an interesting genus–2 Lefschetz fibration constructed by Auroux in [2], which could not be a fiber sum of nontrivial Lefschetz fibrations, did not have any exceptional sections either. This remained as the only known counter-example until recently, where in [4], we obtained several other genus–2 and 3 counter-examples. The purpose of this section is to demonstrate the recipe

\(^1\)That is, the monodromy factorization \(W_{X,f}\), up to Hurwitz equivalence, can be expressed as a product of two nontrivial positive factorizations of 1.
of \[4\] to generate such examples. We will do this while producing quick examples from a well-known relation in the mapping class group. We will then show that Auorux’s example can also be derived in this very scheme. For various background results that goes into this recipe, we advise the reader to turn to \[4\].

### 4.1. Examples derived by monodromy substitutions.

Let \(c_1, \ldots, c_5\) be the simple closed curves on \(\Sigma_2^2\) as shown in Figure 9, and \(\delta_1, \delta_2\) denote the two boundary components with marked points \(u_1, u_2\). The chain relation of length 5

\[(t_{c_1} t_{c_2} t_{c_3} c_{c_4} t_{c_5})^6 = t_{\delta_1} t_{\delta_2}\]

in \(\text{Mod}(\Sigma_2; \{u_1, u_2\})\) (see [10, Proposition 4.12]), prescribes a triple \((X_0, f_0, S_0)\), which is a genus-2 Lefschetz fibration \((X_0, f_0)\) with a pure 2–section \(S\) that consists of two exceptional sections \(S_1, S_2\). It is well-known that \(X_0\) is the K3 surface blown-up twice, the symplectic canonical class of which is represented by \([S] = [S_1] + [S_2]\) in \(H_2(X_0; \mathbb{Z})\).

We will need the following braiding lantern relation, which is a generalization of the lantern relation in the framed mapping class group:

**Lemma 4.1.** [4] Let the curves \(a, b, c, d, x, \delta_1, \delta_2\), pairs of arcs \(y, z\) and points \(u_1, u_2\) in \(\Sigma_6^0\) be as shown in Figure 8 where \(a, b, c, d, \delta_1, \delta_2\) are parallel to boundary components. Denote the boundary components parallel to \(\delta_i\) by \(S_i\). Then the relation

\[t_z t_x t_y = t_a t_b t_c t_d t_{\delta_2}\]

holds in \(\text{Mod}(\Sigma_6^0 \setminus (S_1 \cup S_2); \Sigma_6^0; \{u_1, u_2\})\) (whose elements restrict to identity along the four boundary components without marked points).

![Figure 8. Curves in \(\Sigma_6^0\).](image)

Substituting the subword on the right hand side of the above relation in the framed mapping class group (of a surface that contains the above subsurface with marked boundary components) with the subword on the left is then called a braiding lantern substitution. Its importance and relevance to our current discussion is due to our observation in [4] that whenever the two marked boundary components correspond to two exceptional sections \(S_1, S_2\), we get a new exceptional 2–section \(S_{12}\) after the substitution, which we view as a result of braiding \(S_1\) and \(S_2\) together. Forgetting the two marked boundary components, one gets the usual lantern relation. In this case, the subword \(t_a t_b t_c t_d\) indicates that by clustering the corresponding Lefschetz critical points on the same singular fiber, we can obtain a fiber...
component $V$, which is a sphere of self-intersection $-4$. This $V$, which we will call a lantern sphere, can be assumed to be symplectic with respect to a Gompf-Thurston form. Remembering the two marked boundary components, we conclude that the exceptional classes $S_1$ and $S_2$ each intersect $V$ positively at one point.

Now, one can easily find a lantern sphere $V$ in the monodromy factorization of $(X_0, f_0)$ hit once by each exceptional section $S_i$, $i = 1, 2$. Remarkably, this holds for any lantern sphere (and there are many of them; at least six disjoint ones [1]). This is because for $V$ symplectic, the adjunction equality implies that its intersection with the canonical class $[S_1] + [S_2]$ is 2, whereas each $S_i$ intersects the fibers positively.

Applying the braiding lantern substitution results in a new triple $(X_2, f_2, S_{12})$, a genus–2 Lefschetz fibration with an exceptional 2–section $S_{12}$. As discussed in [4], an observation of Gompf shows that $X_2$ is diffeomorphic to an ordinary blow-down of $X_0$, so $X_2 = K3\#CP^2$, which has only one exceptional class, already represented by $S_{12}$. It follows that $(X_2, f_2)$ does not have any other exceptional sections. On the other hand, it was shown by Usher in [17] that a nonminimal symplectic 4–manifold cannot be a nontrivial fiber sum (also see [3] for a simpler proof for Lefschetz fibrations), so the presence of $S_{12}$ also implies that $(X_2, f_2)$ cannot be a fiber sum of any two nontrivial Lefschetz fibrations. Hence, we have obtained another example of a Lefschetz fibration which cannot arise from Lefschetz pencils via blow-ups and fiber sums.

4.2. Auroux’s genus–2 fibration with an exceptional 2–section.

Let $\delta_1$ and $\delta_2$ be simple closed curves in $\Sigma_2^2$ parallel to the boundary components containing $u_1$ and $u_2$, respectively. We take non-separating simple closed curves $c_1, c_2, c_3, c_4, c_5 \subset \Sigma_2^2$, a path $\gamma \subset \Sigma_2^2$ between $u_1$ and $u_2$ and a pair $\sigma \subset \Sigma_2^2$ of paths connecting the two boundary components as shown in Figure 9. We will denote the right-handed Dehn twist along $c_i$ by $t_i \in \text{Mod}(\Sigma_2^2)$.

**Proposition 4.2.** The following relation holds in $\text{Mod}(\Sigma_2^2; \{u_1, u_2\})$:

$$\tau_1 t_5 t_4 t_3 t_2 t_1 t_1 t_2 t_3 t_5 (t_3 t_2 t_1 t_4 t_3 t_2 t_5 t_4 t_3)^2 t_\sigma = t_{\delta_1}^2 t_{\delta_2}^2.$$

**Proof.** Let $L = t_5 t_4 t_3 t_2 t_1 t_1 t_2 t_3 t_4 t_5$ and $T = t_3 t_2 t_1 t_4 t_3 t_2 t_5 t_4 t_3$. It is easy to see (by the Alexander method (see [10] §. 2.3)), for example) that $t_\sigma$ is equal to $(t_2 t_1)^3 (t_5 t_4)^3 [\iota]$, where $[\iota]$ is the mapping class of the hyperelliptic involution $\iota$ given in Figure 10. Thus, the product $T^2 t_\sigma$ is calculated as follows:

**Figure 9.** Simple closed curves and paths in $\Sigma_2^2$. 


Thus, the product
\[ \tau \text{Mod}(\Sigma_L) \]
\[
\text{is easy to see that the image } \iota \text{ relation holds in } \text{Mod}(\Sigma_L) \text{ in Figure 10 and } \text{Push}(\alpha) \text{ is contained in the image of the homomorphism } \iota_\ast : \text{Mod}(\Sigma^2_L; \{u_1, u_2\}) \to \text{Mod}(\Sigma^2_L; \{u_1, u_2\}) \]
induced by the involution \( \iota \), where \( u \) denotes the point \([u_1] = [u_2] \in \Sigma_L^2 / \iota \) and \( \text{Mod}(\Sigma^2_L; \{u_1, q_2, q_3, q_4, q_5, q_6\}) \) consists of orientation preserving diffeomorphisms fixing \( u, q_1, \ldots, q_6 \) pointwise modulo isotopies fixing the same data. Indeed, it is easy to see that the image \( \iota_\ast(\text{Push}(\alpha)) \) is equal to \( L \), where \( \alpha \) is the loop given in Figure 10 and \( \text{Push}(\alpha) \) is the pushing map along \( \alpha \). In particular the following relation holds in \( \text{Mod}(\Sigma^2_L; \{u_1, u_2\}) \):
\[
L = \tau^{-1}_\gamma t_{\delta_1} t_{\delta_2} [\iota].
\]
Thus, the product \( \tau_\gamma LT^2 \bar{t}_\sigma \) is calculated as follows:
\[
\tau_\gamma LT^2 \bar{t}_\sigma = \tau_\gamma \tau^{-1}_\gamma t_{\delta_1} t_{\delta_2} [\iota] t_{\delta_1} t_{\delta_2} [\iota]
= t_{\delta_1} t_{\delta_2} t_{\delta_1} t_{\delta_2} [\iota] [\iota]
= t_{\delta_1}^2 t_{\delta_2}^2.
\]
This completes the proof. \( \square \)

As we explained in Section 2 we can regard the surface \( \Sigma^2_L \) as a subsurface of \( \Sigma_L \) by capping \( \partial \Sigma^2_L \) by two disks with the centers \( s_1, s_2 \in \Sigma_L \). The pair of paths \( \sigma \) and the path \( \gamma \) respectively give rise to a simple closed curve in \( \Sigma_L \) going through \( s_1 \) and \( s_2 \) and a simple path between \( s_1 \) and \( s_2 \). To simplify the notation, we use the same symbols \( \sigma \) and \( \gamma \) to represent these curves. We also denote the Dehn twist along \( c_i \subset \Sigma_L \) by \( t_i \in \text{Mod}(\Sigma_L; \{s_1, s_2\}) \).
Now by Proposition 4.2, we obtain the following factorization in $\text{Mod}(\Sigma_2; \{s_1, s_2\})$:

$$t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_{10} t_{11} t_{12} t_{13} t_{14} t_{15} t_{16} t_{17} t_{18} t_{19} t_{20} t_{21} t_{22} t_{23} t_{24} t_{25} t_{26} t_{27} t_{28} t_{29} t_{30} = 1,$$

which prescribes a triple $(X_1, f_1, S_1)$ where $f_1 : X_1 \to S^2$ is a genus–2 Lefschetz fibration with a sphere 2–section $S_1$ by Theorem 2.3. Under the forgetful homomorphism $\text{Mod}(\Sigma_2; \{s_1, s_2\}) \to \text{Mod}(\Sigma_2)$ this positive factorization maps to the monodromy factorization of Auroux’s aforementioned genus–2 Lefschetz fibration given in [2]. On the other hand, we can calculate the self-intersection number of $S_1$ using the positive factorization in the framed mapping class group we had in Proposition 4.2 (which of course is a lift of the monodromy factorization $W_{X_1, f_1, S_1}$) and Theorem 1.1 of [4]. Hence $(X_1, f_1, S_1)$ is Auroux’s genus–2 Lefschetz fibration, where $S_1$ is the exceptional 2–section.

We are now ready to show that $(X_1, f_1, S_1)$ can be reproduced using our recipe discussed in the previous subsection. Let $(X_0, f_0, S_0)$ denote the genus–2 Lefschetz fibration with a pure 2–section $S_0$, which is a disjoint union of two exceptional sections, as prescribed by the 5–chain relation $(t_1 t_2 t_3 t_4 t_5)^6 = 1$ in $\text{Mod}(\Sigma_2; \{s_1, s_2\})$.

**Proposition 4.3.** The triple $(X_1, f_1, S_1)$, where $(X_1, f_1)$ is Auroux’s genus–2 fibration with the exceptional 2–section $S_1$, is equivalent to a triple obtained from $(X_0, f_0, S_0)$ by a single braiding lantern substitution, followed by a perturbation of the 2–section.

**Proof.** We first prove that the factorization $(t_1 t_2 t_3 t_4 t_5)^6 = 1$ in $\text{Mod}(\Sigma_2; \{s_1, s_2\})$ is Hurwitz equivalent to the following factorization:

$$t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_{10} t_{11} t_{12} t_{13} t_{14} t_{15} t_{16} t_{17} t_{18} t_{19} t_{20} t_{21} t_{22} t_{23} t_{24} t_{25} t_{26} t_{27} t_{28} t_{29} t_{30} = 1.$$

The factorization (4) can be changed by elementary transformations as follows (in each line elementary transformations are applied to the underlined part to obtain the next line):

$$t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_{10} t_{11} t_{12} t_{13} t_{14} t_{15} t_{16} t_{17} t_{18} t_{19} t_{20} t_{21} t_{22} t_{23} t_{24} t_{25} t_{26} t_{27} t_{28} t_{29} t_{30} = 1,$$

$$t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_{10} t_{11} t_{12} t_{13} t_{14} t_{15} t_{16} t_{17} t_{18} t_{19} t_{20} t_{21} t_{22} t_{23} t_{24} t_{25} t_{26} t_{27} t_{28} t_{29} t_{30} = 1.$$
We take pairs of paths $\xi, \zeta$ and a simple closed curve $a$ as shown in Figure 11. By

\[
\sim t_1 t_2 t_3 t_4 t_5 t_1 t_2 t_3 t_4 t_5 t_3 t_4 t_5 (t_1 t_2 t_3 t_4 t_5)^3
\sim t_1 t_2 t_3 t_4 t_5 t_1 t_2 t_3 t_4 t_5 t_3 t_4 t_5 (t_1 t_2 t_3 t_4 t_5)^3
\sim t_1 t_2 t_3 t_4 t_5 t_1 t_2 t_3 t_4 t_5 t_3 t_4 t_5 (t_1 t_2 t_3 t_4 t_5)^3
\sim t_1 t_2 t_3 t_4 t_5 t_1 t_2 t_3 t_4 t_5 t_3 t_4 t_5 (t_1 t_2 t_3 t_4 t_5)^3
=(t_1 t_2 t_3 t_4 t_5)^6.
\]

Figure 11. Pair of paths and a simple closed curve in $\Sigma_2$.

According to \cite[Remark 3.9]{BaykurHayano2020}, we can perturb the 2–section of the fibration corresponding to $\tilde{t}_\zeta$, and the resulting fibration with a 2–section has the following factorization:

\[
t_3 t_2 t_1 t_4 t_3 t_2 t_5 t_4 t_3 t_5 t_3 t_2 t_3 t_4 t_5 t_3 t_2 t_4 t_5 \tilde{t}_\zeta t_3 t_4 t_5 \tilde{t}_\sigma,
\]

where the equivalence above holds since $t_3 t_2 t_1 t_4 t_3 t_2 t_5 t_4 t_3 t_5 (\xi)$ is equal to $\zeta$.

The last factorization coincides with the monodromy factorization of $(X_1, f_1, S_1)$ in the Equation \cite{BaykurHayano2020} above.

We have thus derived $(X_1, f_1, S_1)$ from $(X_0, f_0, S_0)$, after a braiding lantern substitution, followed by a perturbation of the resulting 2–section to move one of its branched points off the Lefschetz critical locus—while applying Hurwitz moves for Lefschetz fibrations with multisections in various steps of the proof. \hfill $\Box$
References


Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003–9305, USA

E-mail address: baykur@math.umass.edu

Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo, Hokkaido 060-0810, Japan

E-mail address: k-hayano@math.sci.hokudai.ac.jp