MULTISECTIONS OF LEFSCHETZ FIBRATIONS AND MAPPING CLASS GROUP FACTORIZATION

R. İNANÇ BAYKUR AND KENTA HAYANO

Abstract. The purpose of this article is to initiate a study of positive multisections of Lefschetz fibrations via positive factorizations in mapping class groups of surfaces. We show that, using our methods, one can effectively capture various interesting surfaces in symplectic 4-manifolds as multisections, such as Seiberg-Witten basic classes or the curious 2-section of a genus two Lefschetz fibration which leads to a counter-example to Stipsicz’s conjecture on fiber sum indecomposable Lefschetz fibrations.

1. Introduction

Since the groundbreaking work of Donaldson it is known that every symplectic 4-manifold admits a symplectic Lefschetz pencil [5]. Conversely, Gompf showed that every Lefschetz fibration with non-empty critical locus admits a symplectic structure [11]. Thus these results yield a combinatorial description of symplectic 4-manifolds via factorizations of the identity as products of positive Dehn twists in the mapping class group of a closed surface. Here we will extend this fundamental approach by introducing and studying positive factorizations in the mapping class group of a closed surface with framed marked points so as to describe symplectic 4-manifolds together with various important symplectic surfaces in them.

Let $X$ be a closed oriented 4-manifold equipped with a Lefschetz fibration $f: X \to S^2$. We call an embedded, possibly disconnected surface $S$ in $X$ a multisection or $n$-section if $f|_S: X \to S^2$ is an $n$-fold branched cover with only simple branched points. We assume that both Lefschetz critical points and branched points conform to local complex models, that is, we work with positive Lefschetz fibrations and positive branched points. Precise definitions and the basic background material can be found in Section 2 below.

The main result of our article is the description of multisections along with their self-intersection numbers via positive factorizations in mapping class groups of surfaces with framed marked points. This is given in Section 3, where the notion of positivity in this more general setting is introduced as well. The framings amount to working with a new mapping class group of a compact oriented surface with marked boundary circles (one marked point on each boundary component), which consists of isotopy classes of orientation-preserving self-diffeomorphisms that are allowed to swap boundary components while matching the marked points. As we will show, from these positive factorizations, for each multisection one can easily read off the degree (i.e. the number of times it intersects the fibers), topology (number of components and genera of each component) and its self-intersection number.
We shall note that the framed mapping class group we introduce and work with in this article is different than the mapping class group of a surface with boundary which consists of isotopy classes of self-diffeomorphisms that fix each boundary component, the latter being the natural mapping class group to work with when dealing with $n$ disjoint sections. The distinction between these two groups is analogous to that of the framed surface braid group versus the framed pure surface braid group, which have appeared in two recent works that are worth mentioning here: Paolo Bellingeri and Sylvain Gervais studied the exact sequences relating these braid groups [4], whereas Gwénaël Massuyeau, Alexandru Oancea and Dietmar A. Salamon used the same groups to describe the monodromy action of the fundamental group on the first homology of the fiber in terms of the Picard-Lefschetz intersection data associated to vanishing cycles of a given Lefschetz fibration [15].

There are various motivations for our work. Combining the seminal work of Taubes [20] and Donaldson [5], and following the ideas of Donaldson and Smith [6], we observe that a blow-up of any given symplectic 4-manifold $X$ with $b^+(X) > 1$ admits a Lefschetz fibration with respect to which all Seiberg-Witten basic classes are multisections (called standard surfaces in [6]) — see Section 4. Translating this to positive mapping class group factorizations as we prescribed in Section 3, we conclude that symplectic 4-manifolds and their Seiberg-Witten basic classes can be represented combinatorially.

Multisections had yet another major appearance: In [14], Loi and Piergallini showed that any compact Stein surface, up to diffeomorphism, is the total space of an allowable Lefschetz fibration over the 2-disk with bounded fibers, arising as the branched cover of the projection $D^2 \times D^2 \to D^2$ branched along a positive multisection. (Also see [1], where the term PALF was coined for positive allowable Lefschetz fibrations.) It follows that such a multisection along with the fiber surface carries the entire information one needs to describe the diffeomorphism type of any compact Stein surface. In Section 4 we discuss such multisections as examples of rather special multisections which are branched exactly at the Lefschetz critical points of an allowable Lefschetz fibration.

Let us note a further motivation for our study here, which might shed more light on our interest in capturing the self-intersection numbers of multisections. The rather explicit description of a 4-manifold $X$ obtained via monodromy of a Lefschetz fibration on it very often allows one to detect various configurations of symplectic surfaces in it; disjoint copies of fibers and sections, as well as matching pairs of Lefschetz vanishing cycles are a few examples of this sort. Coupled with the non-triviality of Seiberg-Witten invariants on symplectic 4-manifolds, this has been a great source of producing new symplectic and smooth 4-manifolds in the past few decades; see for instance [10] for a survey of such construction methods. A close look at these constructions shows that sections and multisections of such Lefschetz fibrations (which are typically not Seiberg-Witten basic classes) feature a key role. An example of this sort, which is well-known to specialists, is the construction of a curious 6-section in an elliptic Lefschetz fibration with multiplicities by Keum, which plays a key role in the only known topological construction of a fake complex projective plane [13]. We plan to investigate this direction in a future work.
Lastly, in Section 5, we demonstrate how one can use the machinery developed in Section 3 to effectively construct Lefschetz fibrations along with interesting multisections. The first family of examples we provide are the knot sugered elliptic surfaces of Fintushel and Stern, where by exploring new mapping class group relations, we show how to capture all Seiberg-Witten basic classes of these 4-manifolds as multisections of appropriate Lefschetz fibrations on them.

The second example we give in Section 5 is a genus 2 Lefschetz fibration of Auroux which we reconstruct from a positive monodromy factorization together with a 2-section of self-intersection $-1$. This additional topological data on the 2-section then allows us to recapture a theorem of Sato’s [17]: the resulting Lefschetz fibration does not contain any section of self-intersection $-1$, even though it is fiber sum indecomposable — disproving a conjecture of Stipsicz [19].

2. Preliminaries

In this article, we assume that all manifolds are compact, connected, smooth and oriented, and all the maps between them are smooth. Given a map $f : M \to N$ between manifolds $M$ and $N$, we denote by $\text{Crit}(f)$ the set of critical points of $f$.

2.1. Lefschetz fibrations and their multisections.

Let $X$ and $\Sigma$ be compact manifolds (possibly with boundary) of dimensions 4 and 2, respectively.

**Definition 2.1.** A smooth map $f : X \to \Sigma$ is a Lefschetz fibration if $\text{Crit}(f)$ is a discrete set in the interior of $X$ such that for any $p \in \text{Crit}(f)$, we can take a complex coordinate $(U,\varphi)$ (resp. $(V,\psi)$) of $p$ (resp. $f(p)$) compatible with the orientation of $X$ (resp. of $\Sigma$) so that:

$$\psi \circ f \circ \varphi^{-1}(z_1, z_2) = z_1 z_2.$$

We furthermore assume that for each point $q \in C = f(\text{Crit}(f))$, the singular fiber $f^{-1}(q)$ contains exactly one critical point $p \in X$ of $f$. Any point $p \in \text{Crit}(f)$ is called a Lefschetz singularity, and for $g$ the genus of a regular fiber of $f$, $f : X \to \Sigma$ is called a genus-$g$ Lefschetz fibration.

Given any fibration with only Lefschetz critical points, after a small perturbation one can always guarantee that there is at most one critical point on each fiber, as we built into our definition above. It shall be clear that $f$ restricts to a genus $g$ surface bundle over $\Sigma \setminus C$. Lastly, an achiral Lefschetz fibration is defined in the same way as above except that the local coordinate $(U,\varphi)$ is allowed to be incompatible with the orientation of $X$.

Lefschetz fibrations arise naturally from pencils, where the domain 4-manifold is closed and the target surface is $S^2$. A Lefschetz pencil on a closed 4-manifold $X$ is a Lefschetz fibration $f : X \setminus D \to S^2$, defined on the complement of a non-empty discrete set $D$ in $X$, such that around any point $p \in D$, $f$ is locally modeled (again in a manner compatible with orientations) as $(z_1, z_2) \to z_1/z_2$. Blowing-up all the points in $D$, one obtains an honest Lefschetz fibration $\tilde{f} : \tilde{X} \to S^2$ with $|D|$ distinct sections of self-intersection $-1$, namely the exceptional spheres of the respective blow-ups.
Definition 2.2. Let $f : X \rightarrow \Sigma$ be a Lefschetz fibration and $S$ be an embedded surface in $X$. The surface $S$ is called a multisection or $n$-section of $f$ if it satisfies the following conditions:

1. $f|_S$ is an $n$-fold simple branched covering for some non-negative integer $m$;
2. if a branched point $p \in S$ is not in Crit $f$, the induced map $df_p : N_pS \rightarrow T_{f(p)}\Sigma$ is orientation preserving isomorphism, where $N_pS$ is the fiber of the normal bundle of $S$ at $p$ which has the canonical orientation induced by that of $X$ and $S$;
3. if a branched point $p \in S$ of $f|_S$ is in Crit$(f)$, then there are complex coordinates $(U, \varphi)$ and $(V, \psi)$ as in Definition 2.1 above such that $\varphi(S \cap U)$ is equal to $\{(z, z) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}$.

Clearly a 1-section is an honest section of a Lefschetz fibration. Note that in both definitions we have given above, there is a positivity imposed by requiring the compatibility with orientations in local complex models. In the language [6] a multisection which is branched away from Lefschetz singularities is called a standard surface. As it will become clear later (see for instance Section 4.2 below), allowing our multisections to be branched at Lefschetz critical points as well (although subject to the local model given above), we will have a more flexible setting which makes it possible to deal with larger families of examples of Lefschetz fibrations with multisections of geometric significance. Lastly, as in the case of achiral Lefschetz fibrations, one can possibly work more generally with non-necessarily positive multisections by allowing the local models to be incompatible with the orientations.

2.2. Mapping class groups and monodromies of Lefschetz fibrations.

As it will become crucial in capturing the local topology of multisections (namely the self-intersections of them in the ambient 4-manifold), we are going to set up relevant mapping class groups in a framed fashion.

Let $\Sigma$ be a compact, oriented and connected surface. In this paper, we regard $\Sigma$ as the zero-section of the tangent bundle $T\Sigma$. Take subsets $S_i, P \subset T\Sigma$.

We define a group $\text{Mod}_P(\Sigma; S_1, \ldots, S_n)$ as follows:

$$\text{Mod}_P(\Sigma; S_1, \ldots, S_n) = \pi_0(\text{Diff}_P(\Sigma; S_1, \ldots, S_n), \text{id}),$$

where $\text{Diff}_P(\Sigma; S_1, \ldots, S_n)$ is defined as follows:

$$\text{Diff}_P(\Sigma; S_1, \ldots, S_n) = \{T \in \text{Diff}(\Sigma) \mid dT|_P = \text{id}|_P, dT(S_i) = S_i \text{ for } \forall i\}.$$

Here we denote by $\text{Diff}(\Sigma)$ the group of orientation preserving self-diffeomorphisms of $\Sigma$. For simplicity, we also define groups $\text{Diff}^+(\Sigma; S_1, \ldots, S_n)$ and $\text{Mod}(\Sigma; S_1, \ldots, S_n)$ as follows:

$$\text{Diff}^+(\Sigma; S_1, \ldots, S_n) = \text{Diff}_0(\Sigma; S_1, \ldots, S_n),$$

$$\text{Mod}(\Sigma; S_1, \ldots, S_n) = \text{Mod}_0(\Sigma; S_1, \ldots, S_n).$$

The group structures on all of the above are defined via compositions as maps, i.e. for $T_1, T_2 \in \text{Diff}_P(\Sigma; S_1, \ldots, S_n)$, we $T_1 \circ T_2 = T_1 \circ T_2$, etc.

Let $h : X \rightarrow D^2$ be a genus-$g$ Lefschetz fibration and $C = \{p_1, \ldots, p_l\} \subset D^2$ the set of critical values of $h$. We take a regular value $q_0 \in \text{Int}(D^2)$ and an identification $\Sigma_g \cong h^{-1}(q_0)$. For each $i$ we also take a path $\gamma_i$ in $\text{Int}(D^2)$ connecting $q_0$ with $q_i$ so that all $\gamma_i$'s are pairwise disjoint except at $q_0$. We give indices of these paths
so that $\gamma_1, \ldots, \gamma_l$ appear in this order when we travel around $q_0$ counterclockwise. Let $a_i : S^1 \to D^2 \setminus C$ be a loop obtained by connecting a small circle around $q_i$ oriented counterclockwise using $\gamma_i$. The pullback $a_i^* h$ is a $\Sigma_g$-bundle over $S^1$ and we can obtain a self-diffeomorphism by taking a parallel transport of a flow in the total space of $a_i^* h$ transverse to each fiber.

Although a diffeomorphism depends on a choice of a flow, its isotopy class is uniquely determined from the $\Sigma_g$-bundle structure. The isotopy class is called a monodromy of the bundle $a_i^* h$. Kas [12] proved that the monodromy of $a_i^* h$ is the right-handed Dehn twist along some simple closed curve $c_i \subset \Sigma_g$, which is called a vanishing cycle of the Lefschetz singularity $p_i$ in $h^{-1}(q_i)$. Let $a$ be a loop obtained by connecting $a_1, \ldots, a_l$ in this order. It is easy to verify that $a$ is homotopic to the boundary $\partial D^2$ in $D^2 \setminus C$. The product $t_{c_l} \cdots t_{c_1}$ is the monodromy of the bundle $a^* h$. For a genus-$g$ Lefschetz fibration $f : X \to S^2$ over $S^2$, we take a disk $D \subset S^2$ so that $D$ contains all the critical values of $f$. The restriction $f|_{f^{-1}(\partial D)}$ is a Lefschetz fibration over the disk. Since the monodromy of $f|_{f^{-1}(\partial D)}$ is trivial, we can obtain the following factorization of the unit element of the mapping class group $\text{Mod}(\Sigma_g)$:

$$t_{c_l} \cdots t_{c_1} = 1,$$

where $c_i \subset \Sigma_g$ is a vanishing cycle of a Lefschetz singularity of $f$. We call this factorization a monodromy factorization associated with $f$.

3. Multisections via mapping class groups

In this section, we will explain how to deal with multisections of Lefschetz fibrations and self-intersections of them in terms of mapping class groups.

3.1. Local model for the fibration around a regular branched point

Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be the projection onto the first component. We take a subset $S_0 \subset \mathbb{C}^2$ as follows:

$$S_0 = \{(z^2, z) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}.$$ 

The restriction $f_0|_{S_0}$ is a double branched covering branched at the origin.

**Lemma 3.1.** Let $f : X \to \Sigma$ be a Lefschetz fibration, $S \subset X$ a multisection of $f$ and $p \in S \setminus \text{Crit}(f)$ a branched point of $f|_S$. Then, there exist a local coordinate $\Phi : U \to \mathbb{C}^2$ of $p$ and a local coordinate $\varphi : V \to \mathbb{C}$ of $q = f(p)$ which make the following diagram commute:

$$
\begin{array}{ccc}
(U, U \cap S) & \xrightarrow{\Phi} & (\mathbb{C}^2, S_0) \\
\downarrow f & & \downarrow f_0 \\
V & \xrightarrow{\varphi} & \mathbb{C}.
\end{array}
$$

That is, $f_0|_{S_0}$ conforms to a local model of a branched covering map with a simple branched point at $p$.

**Proof.** Since $p$ is not a critical point of $f$, there exist local coordinates $\Phi_0 : U \to \mathbb{C}^2$ and $\varphi_0 : V \to \mathbb{C}$ of $p$ and $f(p)$, respectively, such that $p$ is mapped to the origin of
and that the following diagram commutes:

\[ U \xrightarrow{\varphi_0} \mathbb{C}^2 \]
\[ f \downarrow \quad \downarrow f_0 \]
\[ V \xrightarrow{\psi_0} \mathbb{C}. \]

Without loss of generality, we can assume that the neighborhood \( U \) does not contain any branched points of \( f|_S \) except \( p \). Then, the intersection \( U \cap S \) is diffeomorphic to \( \mathbb{C} \), and \( \Phi_0(S) \) is described as follows:

\[ \Phi_0(S) = \{(s_1(z), s_2(z)) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}, \]

where \( s_i : \mathbb{C} \to \mathbb{C} \) is a smooth function \((i = 1, 2)\).

Since \( p \) is a branched point of \( f|_S \), the map \( s_1 \) is a double branched covering branched at the origin. Thus, there exist diffeomorphisms \( \tilde{\varphi}_1 : \mathbb{C} \to \mathbb{C} \) and \( \varphi_1 : \mathbb{C} \to \mathbb{C} \) which make the following diagram commute:

\[ \mathbb{C} \xrightarrow{\tilde{\varphi}_1} \mathbb{C} \]
\[ s_1 \downarrow \quad \downarrow (\cdot)^2 \]
\[ \mathbb{C} \xrightarrow{\varphi_1} \mathbb{C}. \]

Now, as \( S \) is an embedded surface in \( X \), we can assume that the map \( z \mapsto (s_1(z), s_2(z)) \) is an embedding. In particular, \( s_2 \) is locally diffeomorphic at the origin of \( \mathbb{C} \). Thus, by replacing the local coordinates with sufficiently small ones if necessary, we can take diffeomorphisms \( \tilde{\varphi}_2 : \mathbb{C} \to \mathbb{C} \) and \( \varphi_2 : \mathbb{C} \to \mathbb{C} \) which make the following diagram commute:

\[ \mathbb{C} \xrightarrow{\tilde{\varphi}_2} \mathbb{C} \]
\[ s_2 \downarrow \quad \downarrow \text{id} \]
\[ \mathbb{C} \xrightarrow{\varphi_2} \mathbb{C}. \]

We put \( \Phi_1 = \varphi_1 \times \text{id} \) and \( \Phi_2 = \text{id} \times \varphi_2 \). Now, the following diagram commutes:

\[ U \xrightarrow{\varphi_0} \mathbb{C}^2 \xrightarrow{\varphi_1} \mathbb{C}^2 \xrightarrow{\varphi_2} \mathbb{C}^2 \]
\[ s \downarrow \quad \downarrow f_0 \quad \downarrow f_0 \quad \downarrow f_0 \]
\[ V \xrightarrow{\psi_0} \mathbb{C} \xrightarrow{\varphi_1} \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}. \]

The following equality can be checked easily:

\[ \Phi_2 \circ \Phi_1 \circ \Phi_0(S \cap U) = \{(z^2, \tilde{\varphi}_2 \circ \varphi_1^{-1}(z)) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}. \]

Thus, the diffeomorphisms \( \Phi = (\text{id} \times \tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1}) \circ \Phi_2 \circ \Phi_1 \circ \Phi_0 \) and \( \varphi = \varphi_1 \circ \varphi_0 \) satisfies the desired condition. This completes the proof of Lemma 3.1. \( \square \)

Hence we can always make a local coordinate \( \varphi \) in Lemma 3.1 compatible with the orientation of \( \Sigma \). The branched point \( p \in S \setminus \text{Crit}(f) \) of \( f|_S \) is positive if and only if a local coordinate \( \Phi \) of \( p \) obtained in Lemma 3.1 is compatible with the orientation of \( X \) after making \( \varphi \) compatible with the orientation of \( \Sigma \).
3.2. Standard monodromy factorization around a regular branched point.

We are now going to study the monodromy factorization around a branched point of a multisection, which will play a key role in the proof of Theorem 3.5 below.

We denote by $\Sigma^g$ an oriented, connected and compact surface of genus $g$ with $n$ boundary components. Let $S_0 \subset \mathbb{C}^2$ be a standard model of a branched point away from Lefschetz singularities as explained in the previous subsection. Denote by $B_k$ the subset $\{z \in \mathbb{C} \mid |z| \leq k\}$. We consider the restriction $q = p|_{B_1 \times B_2} : B_1 \times B_2 \rightarrow B_1$. The subset $S_0 \cap (B_1 \times B_2)$ is a bisection of $q$. This bisection, together with an identification $B_2 \cong \Sigma^1_0$, makes the monodromy $g_0$ of $q|_{q^{-1}(\partial B_1)}$ be contained in the group $\text{Mod}_{\partial \Sigma^1_0}(\Sigma^1_0; \{s_1, s_2\})$ where $s_1, s_2$ are two points in $q^{-1}(1) \cap S_0$. It is known that this monodromy is equal to the positive half twist along an arc between $s_1$ and $s_2$. Let $\varepsilon \in \mathbb{R}$ be a sufficiently small real number and we put $e = 1 - \varepsilon$. We take another subset $S'_0 \subset \mathbb{C}^2$ as follows:

$$S'_0 = \{(z^2, ez) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}.$$

The subset $S'_0 \cap (B_1 \times B_2)$ is also a bisection of $q$. By using the bisections, we can take a lift $\tilde{g}_0 \in \text{Mod}_{\partial \Sigma^3_0}(\Sigma^3_0; \{u_1, u_2\})$ of the monodromy $g_0$, where $u_1, u_2$ are points in $\partial \Sigma^3_0 \setminus \partial \Sigma^1_0$ which cover the set $\pi_0(\partial \Sigma^3_0 \setminus \partial \Sigma^1_0)$. Note that the group $\text{Mod}_{\partial \Sigma^3_0}(\Sigma^3_0; \{u_1, u_2\})$ is isomorphic to the group $\text{Mod}_{\partial \Sigma^3_0}(\Sigma^3_0; \{v_1, v_2\})$, where $v_i$ is a non-zero tangent vector in $\Sigma^3_0$.

**Lemma 3.2.** The element $\tilde{g}_0$ is equal to the element described as in Figure 1.

![Figure 1](image-url)  

**Figure 1.** The element $\tilde{g}_0$ interchanges the points $u_1, u_2$, and keeps the dotted arc between $u_1$ and $u_2$.

**Proof.** The element $\tilde{g}_0$ is a lift of $g_0$. Thus, the bold arc in Figure 1 is mapped as described in the figure. It is sufficient to prove an arc connecting $s_1$ and $s_2$ is preserved by some representative of $\tilde{g}_0$ since the group $\text{Mod}_{\partial \Sigma^3_0}(\Sigma^3_0; \{u_1, u_2\})$ is isomorphic to the group $\text{Mod}_{\partial \Sigma^3_0}(\Sigma^3_0; \{v_1, v_2\})$. We take an arc $\gamma \subset q^{-1}(1)$ as follows:

$$\gamma = \{(1, (1 - 2t)) \in \mathbb{C}^2 \mid t \in [0, 1]\}.$$ 

This arc connects the two points in $S_0 \cap q^{-1}(1)$. We take a horizontal distribution $\mathcal{P}$ of $q|_{q^{-1}(\nu \partial B_1)}$ so that it coincides the following distribution on $\partial B_1 \times B_2$:

$$\langle \left( \frac{\partial}{\partial x_1} + \frac{x_3}{2} \frac{\partial}{\partial x_3} \right) - \frac{x_4}{2} \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} + \frac{x_4}{2} \frac{\partial}{\partial x_3} + \frac{x_3}{2} \frac{\partial}{\partial x_4} \rangle.$$ 


where \((x_1, x_2, x_3, x_4)\) is a real coordinate determined by the formula \((z_1, z_2) = (x_1 + \sqrt{-1}x_2, x_3 + \sqrt{-1}x_4)\). We define a loop \(c : [0, 2\pi] \to \partial B_1\) as follows:

\[c(t) = \exp(\sqrt{-1}t)\]

Take a point \(t_0 \in [-1, 1]\). It is easy to see that the horizontal lift \(\tilde{c}_{t_0}(t)\) with base point \(w = (1, 0, t_0, 0) \in q^{-1}(1)\) is given by:

\[
\tilde{c}_{t_0}(t) = \left(\cos(t), \sin(t), t_0 \cos\left(\frac{t}{2}\right), t_0 \sin\left(\frac{t}{2}\right)\right).
\]

Thus, the arc \(\gamma\) is preserved by the parallel transport along the curve \(c\) with respect to \(P\). Since this parallel transport is a representative of \(\tilde{\rho}_0\), this completes the proof of Lemma 3.2. \(\square\)

The two bisections \(S_0\) and \(S'_0\) intersect only at the origin, but do not intersect transversely. In order to make the two bisections intersect transversely, we will take a small perturbation of \(S'_0\). We first take a smooth function \(\rho : \mathbb{R} \to [0, \varepsilon]\) satisfying the following conditions:

(a) \(\rho(t) = \rho(-t)\);
(b) \(\rho(t) = \varepsilon^2\) for all \(t \in [0, \frac{\varepsilon}{2}]\);
(c) \(\rho(t) = 0\) for all \(t \in [\varepsilon, \infty)\);
(d) \(-3\varepsilon < \frac{d\rho}{dt}(t) < 0\) for all \(t \in [\frac{\varepsilon}{2}, \varepsilon]\).

We define the subset \(S'_{0, \rho} \subset \mathbb{C}^2\) as follows:

\[S'_{0, \rho} = \{(z^2, ez + \rho(|z|^2)) \in \mathbb{C}^2 | z \in \mathbb{C}\}.
\]

The two subsets \(S_0\) and \(S'_{0, \rho}\) intersect at the two points \((r_1^2, r_1), (r_2^2, r_2) \in \mathbb{C}^2\), where \(r_1, r_2 \in \mathbb{R}\) is the real numbers which satisfy the following conditions:

\[r_1 = \frac{\rho(r_1^2)}{\varepsilon}, \quad r_2 = \frac{\rho(r_2^2)}{2 - \varepsilon}.
\]

It is easy to see that \(S_0\) intersects \(S'_{0, \rho}\) at the two points transversely, and that both signs of these intersections are positive with respect to the standard orientation of \(\mathbb{C}^2\).

3.3. Multisections branching at Lefschetz critical points.

We will now study the local model around a branched point of a multisection coinciding with a Lefschetz critical point of the fibration. We take two points \(s_1, s_2 \in \text{Int}(\Sigma_0^2)\). We denote by \(i : \Sigma_0^2 \to \Sigma_0^2\) the hyperelliptic involution with fixed point set \(\{s_1, s_2\}\). We define a group \(\mathcal{H}(\Sigma_0^2)\) as follows:

\[\mathcal{H}(\Sigma_0^2) = \pi_0(\text{Diff}_0^+(\Sigma_0^2), \iota, \text{id}),\]

where \(C(\Sigma_0^2, \iota) = \{\varphi \in \text{Diff}_0^+(\Sigma_0^2) | \iota \circ \varphi = \varphi \circ \iota\}\). The quotient space \(\Sigma_0^2/\iota\) is diffeomorphic to the disk \(\Sigma_0^1\). Denote by \(s'_1, s'_2\) corresponding points in \(\Sigma_0^1\) with \(s_1, s_2\), respectively. The group \(\text{Mod}_{\Sigma_0^2}^{\Sigma_0^1}(\Sigma_0^1; \{s'_1, s'_2\})\) is an infinite cyclic group. The natural map \(\mathcal{H}(\Sigma_0^2) \to \text{Mod}_{\Sigma_0^2}^{\Sigma_0^1}(\Sigma_0^1; \{s'_1, s'_2\})\) induced by a quotient map is injective. The inclusion map \(C(\Sigma_0^2, \iota) \to \text{Diff}_0^+(\Sigma_0^2)\) induces the homomorphism \(i : \mathcal{H}(\Sigma_0^2) \to \text{Mod}_{\Sigma_0^2}^{\Sigma_0^1}(\Sigma_0^1) \cong \mathbb{Z}\). Since this map is surjective, the group \(\mathcal{H}(\Sigma_0^2)\) is also an infinite cyclic group and the map \(i\) is an isomorphism. On the other hand,
the inclusion map $C(\Sigma^2_0, t) \hookrightarrow \text{Diff}^+_\partial\Sigma^2_0(\Sigma^2_0; \{s_1, s_2\})$ also induces a homomorphism $\iota_*: \mathcal{H}(\Sigma^2_0) \rightarrow \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\})$. We denote by
\[
F_{s_1, s_2} : \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\}) \rightarrow \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0)
\]
the forgetful map. Since the composition $F_{s_1, s_2} \circ \iota_*$ is equal to $\iota$ and $\iota$ is isomorphic, the map $\iota_*$ is injective. Thus, we can regard the group $\mathcal{H}(\Sigma^2_0) \cong \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\})$. Under this identification, the Dehn twist $t_c \in \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0)$ along the curve parallel to $\partial\Sigma^2_0$, which is the generator of this group, is regarded as an element in $\text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\})$ described in Figure 2.

![Figure 2. The element $t_c$ interchanges $s_1$ and $s_2$.](image)

We denote by $Y \subset \mathbb{C}^2$ the intersection $B_2 \times B_2 \cap f^{-1}(B_1)$, where $B_k$ is the disk $\{z \in \mathbb{C} \mid |z| \leq k\}$ and $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ is the standard local model of a Lefschetz singularity, that is, $f$ is defined as $f(z_1, z_2) = z_1 z_2$. We also denote by $f_0$ the restriction $f|_Y$. The regular fiber $f_0^{-1}(1)$ is the annulus $\Sigma^2_0$ and we fix an identification $f_0^{-1}(1) \cong \Sigma^2_0$. Take the standard bisection of $\Delta_0 = \{(z, z) \in Y \mid z \in B_1\}$ of $f_0$. We define the involution $\eta : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ as follows:
\[
\eta(z_1, z_2) = (z_2, z_1).
\]
The fixed point set of $\eta$ is equal to $\Delta_0$. We take an identification $f_0^{-1}(1) \cong \Sigma^2_0$ so that the restriction $\eta|_{f_0^{-1}(1)}$ equals to the involution $\iota$. By taking a horizontal distribution $\mathcal{P}$ of the fibration $f_0|_{Y \setminus \{0\}}$ which is along both $\Delta_0$ and $\partial Y$, we can regard the monodromy $\eta_0$ of $\partial B_1$ as an element of the group $\text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\})$, where $\{s_1, s_2\}$ is the intersection $\Delta_0 \cap f_0^{-1}(1)$.

**Lemma 3.3.** Under the identification of $f_0^{-1}(1)$ with $\Sigma^2_0$ as above, the monodromy $\eta_0$ is equal to the Dehn twist $t_c \in \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\})$.

**Proof.** We take a horizontal distribution $\mathcal{P}$ so that $\mathcal{P}$ is preserved by $\eta$. The monodromy $\eta_0$ is contained in the group $\mathcal{H}(\Sigma^2_0) \subset \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\})$. Furthermore, using the result in [12], we can prove that this monodromy is mapped to the Dehn twist $t_c \in \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0)$ by $F_{s_1, s_2}$. This completes the proof of Lemma 3.3. \(\square\)

We take a disk neighborhood $D_i \subset \Sigma^2_0$ of the point $s_i$ which is preserved by $\iota$. We put $D = D_1 \amalg D_2$ and fix an identification $\Sigma^2_0 \setminus D \cong \Sigma^4_0$. We also take a point $u_i \in \partial D_i$. We can define the following homomorphism:
\[
\text{Cap} : \text{Mod}_{\partial\Sigma^2_0}(\Sigma^4_0; \{u_1, u_2\}) \rightarrow \text{Mod}_{\partial\Sigma^2_0}(\Sigma^2_0; \{s_1, s_2\}),
\]
by capping $\Sigma^4_0$ by $D$. 
We take a sufficiently small number $\varepsilon > 0$ and put $\xi = \exp(\sqrt{-1}\varepsilon)$. We define another bisection $\Delta'_0$ of $f_0$ as follows:

$$\Delta'_0 = \{(\xi z, \xi^{-1} z) \in Y \mid z \in B_1\}.$$ 

Note that $\Delta'_0$ intersects $\Delta_0$ at the origin transversely. This bisection, together with the bisection $\Delta_0$, gives a lift $\tilde{\varrho}_0 \in \text{Mod}_{\partial \Sigma}^\infty(\Sigma_4^3; \{u_1, u_2\})$ of the monodromy $\varrho_0$ under the map $\text{Cap}$.

**Lemma 3.4.** Under a suitable identification $\Sigma_4^3 \cong f_0^{-1}(1) \setminus \nu \Delta_0$, the monodromy $\tilde{\varrho}_0$ is equal to the element described in Figure 3.

**Proof.** We define a subset $C(\Sigma^4_0, \iota)$ of $\text{Diff}^+_{\partial \Sigma}^\infty(\Sigma_0^4, \{u_1, u_2\})$ as follows:

$$C(\Sigma^4_0, \iota) = \{T \in \text{Diff}^+_{\partial \Sigma}^\infty(\Sigma_0^4, \{u_1, u_2\}) \mid T \circ \iota = \iota \circ T\}.$$ 

Note that the mapping class described in Figure 3 is contained in the group $\pi_0(C(\Sigma^4_0, \iota), \text{id})$. By the same argument as in the proof of Lemma 3.3, we can assume that the element $\tilde{\varrho}_0$ is contained in the group $\pi_0(C(\Sigma^4_0, \iota), \text{id})$. It is easy to see that the following map is a diffeomorphism:

$$\begin{align*}
\mathbb{C}^2 / \eta &\rightarrow \mathbb{C}^2 \\
\{(z_1, z_2)\} &\mapsto \left(z_1 z_2, \frac{z_1 + z_2}{2}\right).
\end{align*}$$

We identify these spaces via this diffeomorphism. The following diagram commutes:

$$\begin{array}{ccc}
(C^2, \Delta_0, \Delta'_0) & \xrightarrow{\eta} & (C^2, S_0, S'_0) \\
\downarrow f & & \downarrow p \\
\mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C},
\end{array}$$

where $S_0$ and $S'_0$ are the subsets of $\mathbb{C}^2$ defined in the previous section (in this case, $\varepsilon$ is equal to $\text{Re}(\xi)$). Thus, the monodromy $\tilde{\varrho}_0$ is mapped to the element described in Figure 1 by the natural injection $\pi_0(C(\Sigma^4_0, \iota), \text{id}) \hookrightarrow \text{Mod}_{\partial \Sigma}^\infty(\Sigma_0^3, \{u_1, u_2\})$. On the other hand, it is easy to verify that the mapping class described in Figure 3 is also mapped to that described in Figure 1 by the same injection. This completes the proof of Lemma 3.4. \qed
3.4. Capturing multisections via mapping class group factorizations.

With all the preliminary results we have obtained in the previous subsections, we are now ready to prove the main theorem of our paper:

**Theorem 3.5.** Let \( f : X \to S^2 \) be a genus-\( g \) Lefschetz fibration, and \( t_{c_1} \cdots t_{c_l} = 1 \) be a monodromy factorization of \( f \). Suppose that \( S \subset X \) is an \( n \)-section of \( f \) with \( k \) branched points away from \( \text{Crit}(f) \), and \( r \) branched points on the Lefschetz singularities of \( f \) at Lefschetz singularities corresponding to cycles \( c_1, \ldots, c_r \), \( r \leq l \). Denote by \( m \) the self-intersection of the multisection \( S \). Then, there exists a lift \( \tilde{c}_i \subset \Sigma^g_0 \) of \( c_i \) such that the following equation holds in \( \text{Mod}(\Sigma^g_0; \{u_1, \ldots, u_n\}) \):

\[
\tilde{t}_1 \cdots \tilde{t}_k \cdot \tilde{c}_{c_1} \cdots \tilde{c}_{c_t} = t_1^{a_1} \cdots t_p^{a_p},
\]

where \( \{u_1, \ldots, u_p\} \) is a subset of \( \partial \Sigma^g_0 \) which covers all the elements of \( \pi_0(\partial \Sigma^g_0) \), \( \tilde{t}_i \)
and \( \tilde{c}_i \) are lifts of a half twist along an arc between \( s_{1_i}, s_{2_i} \) in \( \text{Mod}(\Sigma^g_0; \{s_1, \ldots, s_p\}) \) as described in Figure 1, \( \tilde{c}_i \) is a lift of the Dehn twist \( t_{c_i} \) as described in Figure 3, \( \{\delta_1, \ldots, \delta_p\} \) is a set of simple closed curves parallel to \( \partial \Sigma^g_0 \), and \( a_i \) is an integer satisfying the condition \( m = -\Sigma_{i=1}^n a_i + 2k + r \).

Conversely, for any relation in \( \text{Mod}(\Sigma^g_0; \{u_1, \ldots, u_n\}) \) of the form (1) above, there exists a genus-\( g \) Lefschetz fibration \( f : X \to S^2 \) with a \( n \)-fold section \( S \subset X \) of self-intersection \( m = -\Sigma_{i=1}^n a_i + 2k + r \) whose monodromy factorization is given by the image of the factorization in the left hand side of (1) under the map \( i : \text{Mod}(\Sigma^g_0; \{u_1, \ldots, u_n\}) \to \text{Mod}(\Sigma_g) \) which is induced by the inclusion \( i : \Sigma^g_0 \hookrightarrow \Sigma_g \).

Note that after relabeling the arcs we choose for the monodromy description of \( f \), we can always assume that the first \( r \) cycles are the ones corresponding to those where \( S \) is branched at.

**Proof.** For a given genus-\( g \) Lefschetz fibration \( f : X \to S^2 \) with a \( n \)-fold section \( S \) and its monodromy factorization \( t_{c_1} \cdots t_{c_l} = 1 \), let \( \gamma_1, \ldots, \gamma_l \) be reference paths from a regular value \( q_0 \in S^2 \) which gives the factorization \( t_{c_1} \cdots t_{c_l} = 1 \). We take reference paths \( \alpha_1, \ldots, \alpha_k \) satisfying the following properties:

- \( \alpha_i \) connects \( p_0 \) with the image of a positive branched point of \( S \);
- \( \alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_{l-k} \) appear in this order when we go around \( q_0 \) counterclockwise.

We take a perturbation \( S' \) of \( S \) so that the pair \((S, S')\) coincides with either of the pairs \((S_0, S_0')\) or \((\Delta_0, \Delta_0')\) in a small coordinate neighborhood of each branched point of \( S \). This perturbation gives a lift of monodromies of \( f \) to the group \( \text{Mod}(\Sigma^g_0; \{u_1, \ldots, u_n\}) \). By Lemma 3.2, local monodromies obtained from paths \( \alpha_i \) are lifts of half twists described in Figure 1 and their inverse, respectively. On the other hand, by Lemma 3.4, a local monodromy obtained from a path \( \gamma_i \) (\( i \in \{1, \ldots, l-k\} \)) is a lift of the Dehn twist \( t_{c_i} \) described in Figure 3. Thus, we can obtain a factorization as in Theorem 3.5.

Using the observation following the proof of Lemma 3.2 and the fact that \( \Delta_0 \) intersects \( \Delta_0' \) at the origin transversely, it is easy to verify that this factorization satisfies the condition on self-intersection of \( S \).
Conversely, for a given lift of a factorization as in Theorem 3.5, we can easily construct a genus-\( g \) Lefschetz fibration \( f : X \to S^2 \) and a \( n \)-fold section \( S \) of \( f \) with desired conditions by pasting local models given in the present section according to the factorization. The details are left to the readers. \( \square \)

**Remark 3.6.** After a small modification of the proof above we can similarly obtain a monodromy factorization for a *not necessarily positive* multisection, where each negative branched point away from \( \text{Crit}(f) \) contributes \(-2\) and each branched point at a negative critical point contributes \(-1\) to the total count of the self-intersection of the multisection.

**Remark 3.7.** We shall note that, although multisections going through Lefschetz critical points are of particular interest in certain contexts (see for instance Section 4.2 below), it is in fact always possible to perturb any given multisection of a Lefschetz fibration so as to obtain one which is branched completely away from the Lefschetz critical points. This can be achieved by the following perturbation around each branched point on a Lefschetz singularity:

\[
\Delta_\epsilon = \{(z + \epsilon, z - \epsilon) \in \mathbb{C}^2 \mid z \in \mathbb{C}\},
\]

where \( \epsilon \) is a sufficiently small positive number. In this perturbation, a branched point on a Lefschetz singularity is substituted for a positive branched point. Indeed, we can easily verify the following relation in the group \( \text{Mod}_{\omega_{S^2}}(\Sigma^4_0; \{u_1, u_2\}) \) (using the Alexander method [7, Proposition 2.8], for example):

\[
\tilde{t}_c = t^{-1}_{c'} t_{\tau},
\]

where \( \delta \) is a simple closed curve parallel to the boundary component containing \( u_1 \), \( c' \) is a simple closed curve described in Figure 4 and \( \tilde{\tau} \) is a lift of a half twist preserving the path \( \tau \) in Figure 4. Using Theorem 3.5, we can then make a multisection avoid Lefschetz singularities by substituting a lift \( \tilde{t}_c \) in a lift of a factorization (1) for the right side of (2).

4. General examples: Seiberg-Witten basic classes, Stein surfaces and positive multisections

Here we would like to discuss how multisections play a key role in two important contexts, as representatives of Seiberg-Witten basic classes of closed symplectic 4-manifolds, and as the building data of compact Stein surfaces via Lefschetz pencils/fibrations they are equipped with, respectively. We shall note that these observations build on earlier work of Taubes, Donaldson-Smith, and Loi-Piergallini, our goal here is to simply point out how they relate to our study of multisections.
4.1. Seiberg-Witten basic classes of symplectic 4-manifolds.

Let $X$ be a symplectic 4-manifold with $b^+(X) > 1$. We further assume that it has an integral symplectic form $\omega$, which can always be achieved by replacing a given form with a multiple of a rational symplectic form approximating it. By Taubes, for a generic almost complex structure $J$ on $(X, \omega)$, any Seiberg-Witten basic class $\beta \in H_2(X; \mathbb{Z})$ can be represented by a sum of $J$-holomorphic curves $C_i$ in $X$ [20]. Moreover, each component of the representative of $\beta = \sum_i [C_i]$ is an embedded smooth curve unless it is a torus of self-intersection zero (in which case the image of the curve is still smoothly embedded, but the parametrization is a multiple cover) or a sphere of negative self-intersection. Since $J$ is $\omega$ tamed, each $C_i$ is a symplectic surface in $(X, \omega)$.

Since the number of basic classes of a 4-manifold is finite, so is the collection of the symplectic surfaces $C_i$, sums of which represent the basic classes in $(X, \omega)$. As noted by Donaldson and Smith [6, Proposition 2.9] replacing $\omega$ with a sufficiently high multiple $k\omega$, we can then assume that there exists a symplectic Lefschetz pencil on $X$ for which all $C_i$ are multisections (“standard surfaces” in the language of [6]). By the blow-up formula for Seiberg-Witten classes, we conclude that after passing to a blow-up of $X$ we get a symplectic Lefschetz fibration $f : \tilde{X} \to S^2$ where all basic classes are represented by a collection of symplectic surfaces $C_i$ and the exceptional spheres $E_j$. Hence, each Seiberg-Witten basic class of $\tilde{X}$ is represented by a multisection (possibly with several components).

To sum up, combining the works of Taubes and Donaldson, after passing to a blow-up $\tilde{X}$, one can represent all Seiberg-Witten classes of a symplectic 4-manifold $X$ as multisections with respect to a Lefschetz fibration.

We shall note that this is merely an existence result, as the construction of such a Lefschetz fibration is not explicit. For examples of Lefschetz fibrations on symplectic 4-manifolds along with multisections representing their basic classes, the reader shall visit the very last section of our paper, where we provide explicit examples using new monodromy factorizations capturing multisections.

4.2. Stein surfaces as allowable Lefschetz fibrations.

Here we will review a fundamental result of Loi and Piergallini in order to illustrate how multisections branched at Lefschetz critical points naturally arise in the context of allowable Lefschetz fibrations on compact Stein surfaces. Here the extra adjective “allowable” means that all Lefschetz cycles are homologically essential on the fiber, and the Lefschetz fibration in question naturally has fibers with non-empty boundary.

In [14], Loi and Piergallini showed that up to orientation-preserving diffeomorphisms any compact Stein surface can be realized as the total space of an allowable Lefschetz fibration. Their proof generalizes the pioneering work of Rudolph, who showed that the intersection of any non-singular complex curve with the unit disk $D^4$ in $\mathbb{C}^2$ can be realized as a braided surface $S \subset D^4$ that in our language lies as a multisection of the natural fibration $\pi : D^4 \to D^2$ [16], restricting the boundary $S^3 = \partial D^4$ as a quasipositive link. The authors proved that in general any compact Stein Surface $X$ can be realized as a branched cover $g : X \to D^4$ branched along a braided surface $S$ in $D^4$, so that the composition $f = \pi \circ g : X \to D^2$ is an allowable Lefschetz fibration, and vice versa. Roughly speaking, this result shows
that allowable Lefschetz fibrations are to compact Stein manifolds what Lefschetz pencils are to closed symplectic 4-manifolds, which since then played a heroic role in the study of Stein fillings of contact 3-manifolds.

The key point in Loi and Piergallini’s construction of the allowable Lefschetz fibration in question is that each branched point of $S$ in $D^4$ gives rise to a branched point in $X$ at a Lefschetz critical point, and $\text{Crit}(f)$ above at the end consists of only these points. Thus, the branched surface $S_X \cong S$ in $X$ is an $n$-section of the allowable Lefschetz fibration $f : X \to D^4$, where $n$ is the degree of the underlying branched covering $g : X \to D^4$ and the branching data completely recaptures the topology of $X$ as well as the Stein structure obtained on it by pulling-back the canonical Stein structure on $D^4$.

5. Examples of multisections

In this section, we will present various mapping class group factorizations which will allow us to capture important families of examples of Lefschetz fibrations along with their multisections.

5.1. Knot surgered elliptic surfaces and their Seiberg-Witten basic classes.

Let $X$ be a smooth 4-manifold and $T \subset X$ an embedded torus with self-intersection 0. For a knot $K \subset S^3$, we denote by $M_K$ the knot complement $S^3 \setminus \nu K$. A knot surgery 4-manifold $X_K$ is defined as follows:

$$X_K = X \setminus \nu T \cup S^1 \times M_K.$$

Fintushel and Stern [8] introduced this operation and proved that a Laurent polynomial associated with the Seiberg-Witten invariant of $X_K$ is the product of that of $X$ and the symmetrized Alexander polynomial of the knot $K$ under some assumption on $T \subset X$. If $T$ is a regular fiber of the elliptic fibration $E(n)$ and $K$ is a fibered knot with genus-$g$, the knot surgery 4-manifold $E(n)_K$ admits a genus-$(2g + n - 1)$ Lefschetz fibration (see [9] for details). We will see bisections of this fibration corresponding to regular fibers of the elliptic fibration on $E(n)$ in terms of mapping class groups.

Let $A_1, \ldots, A_{2g-2}, B_1, \ldots, B_{2g+1}, C_1, C_2$ be simple closed curves in $\Sigma_{2g+n-1}$ as described in Figure 5. We remove two disks $D_1, D_2$ from $\Sigma_{2g+n-1}$ as in Figure 5 and take points $u_1, u_2$ on each boundary component of $\Sigma_{2g+n-1} = \Sigma_{2g+n-1} \setminus (D_1 \cup D_2)$. Let $K$ be a fibered knot with genus-$g$ and $\varphi_K \in \text{Mod}(\Sigma_g)$ a monodromy of $K$. We decompose $\Sigma_{2g+n-1}$ into three pieces: the upper $\Sigma_g$, the lower $\Sigma_g$ and the central $\Sigma_{n-1}$ in Figure 5, so that both of the disks $D_1$ and $D_2$ are contained in $\Sigma_{n-1}$. Let $\Phi_K$ be an element $\text{Mod}(\Sigma_{2g+n-1})$ defined as follows:

$$\Phi_K = \varphi_K \# \text{id} \# \text{id} : \Sigma_g \# \Sigma_{n-1} \# \Sigma_g \to \Sigma_g \# \Sigma_{n-1} \# \Sigma_g.$$

The genus-$(2g + n - 1)$ Lefschetz fibration $f_{n,K} : E(n)_K \to S^2$ mentioned above has the following monodromy factorization (see [9]):

$$\eta_{n,g} \eta_{n,g} \Phi_K(\eta_{n,g}) \Phi_K(\eta_{n,g}) = 1,$$

where $\eta_{n,g}$ is equal to $t_{A_{2g-2}} \cdots t_{A_1} t_{A_1}^{-1} \cdots t_{A_{2g-2}} t_{B_0} \cdots t_{B_{2g+1}}$ and $\Phi_K(\eta_{n,g})$ is a factorization obtained from $\eta_{n,g}$ by substituting $A_i$ and $B_j$ in $\eta_{n,g}$ for $\Phi_K(A_i)$ and $\Phi_K(B_j)$, respectively.
**Figure 5.** Simple closed curves and a path in \( \Sigma_{2g+n-1}^2 \).

**Proposition 5.1.** The following equality holds in \( \text{Mod}(\Sigma_{2g+n-1}^2; \{u_1, u_2\}) \):

\[
t_{A_{2n-2}} \cdots t_{A_1} t_{A_{2n-2}} t_{B_n} \cdots t_{B_{2g+1}} = t_\delta t_\delta^{-1} t_\iota,
\]

where \( \delta \) is a simple closed curve in \( \Sigma_{2g+n-1}^2 \) parallel to the boundary component containing \( u_1 \), \( \bar{\tau} \) is a lift of a half twist along a path \( \tau \) in Figure 5 as described in Figure 1 and \( \iota \) is an involution described in the left side of Figure 5.

**Proof.** We cut the surface \( \Sigma_{2g+n-1}^2 \) along the curves \( C_1, C_2 \) to obtain the surface \( \Sigma_{2g}^4 \). Take points \( u_3 \in C_3 \) and \( u_4 \in C_2 \). Denote by \( v_1, v_2 \) the fixed points of \( \iota \) in \( \Sigma_{2g}^4 \) and \( \tau_i \) along the path \( \tau \) in Figure 5 as described in Figure 1.

We define \( C(\Sigma_{2g}^4; \iota) \) as follows:

\[
C(\Sigma_{2g}^4; \iota) = \{ \varphi \in \text{Diff}^+(\Sigma_{2g}^4; \{u_1, u_2, u_3\}) \mid \iota \circ \varphi = \varphi \circ \iota \}.
\]

Since the simple closed curve \( B_i \) is preserved by \( \iota \), we can regard the Dehn twist \( t_{B_i} \) as an element in \( \tau_0 C(\Sigma_{2g}^4; \iota), \text{id} \). The quotient space \( \Sigma_{2g}^4 / \iota \) is diffeomorphic to \( \Sigma_{2g}^2 \). Thus, the quotient map \( \iota_* : \pi_0(\Sigma_{2g}^4; \iota), \text{id} \to \text{Mod}(\Sigma_{2g}^2; \{u_1, u_3\}, \{v_1, v_2\}) \).

Let \( \kappa \) be an involution of \( \Sigma_g^2 \) as described in the middle of Figure 6. We define \( C(\Sigma_g^2; \kappa) \) as follows:

\[
C(\Sigma_g^2; \kappa) = \{ \varphi \in \text{Diff}^+(\Sigma_g^2; \{u_1, u_3\}) \mid \kappa \circ \varphi = \varphi \circ \kappa \}.
\]
where \( \theta \) is an element in \( \pi_0(C(\Sigma^2_g; \kappa), \text{id}) \). Since the quotient space \( \Sigma^2_g/\kappa \) is diffeomorphic to \( \Sigma^1_0 \), the quotient map \( /\kappa : \Sigma^2_g \to \Sigma^1_0 \) induces the following map:

\[
\kappa_\ast : \pi_0(C(\Sigma^2_g; \kappa), \text{id}) \to \text{Mod}_{\partial \Sigma^1_0}(\Sigma^1_0; \{w_0, \ldots, w_{2g+1}\}),
\]

where \( w_0, \ldots, w_{2g+1} \) are the images of the branched points of \( /\kappa : \Sigma^2_g \to \Sigma^1_0 \). Note that the kernel of \( \kappa_\ast \) is generated by the isotopy class of \( \kappa \). Let \( B''_0 \) be the image of \( B'_0 \) under \( /\kappa \) (see Figure 6). We take an oriented loop \( \beta_t \subset \Sigma^1_0 \) based at \( v' = /\kappa(v_1) \) by connecting \( p_0 \) with a small circle around \( u_1 \) oriented counterclockwise using \( B''_0 \). The following equality holds in \( \text{Mod}_{\partial \Sigma^1_0}(\Sigma^1_0; \{w_0, \ldots, w_{2g+1}\}) \):

\[
\kappa_\ast(\tau_0 \cdots \tau_{2g+1}) = \text{Push}(\beta_0) \cdots \text{Push}(\beta_{2g+1}) = \text{Push}(\mu),
\]

where \( \mu \) is an oriented based loop described in Figure 6. Combining the equalities (3) and (4), we obtain the following equality in \( \text{Mod}(\Sigma^4_g; \{u_1, u_2, u_3, u_4\}) \):

\[
l_{B_0} \cdots l_{B_{2g+1}} = \tilde{\tau}^{-1} \tilde{\theta}^{-1} l_{C_1} l_{C_2} l_{\delta_1} l_{\delta_2} l,
\]

where \( \theta \subset \Sigma^4_g \) is a path between \( u_3 \) and \( u_4 \) described in Figure 7. Thus, the product \( l_{A_{2n-2}} \cdots l_{A_1} l_{A_{2n-2}} l_{B_0} \cdots l_{B_{2g+1}} \) is calculated as follows:

\[
=t_{A_{2n-2}} l_{A_1} l_{A_{2n-2}} l_{B_0} \cdots l_{B_{2g+1}}
\]

\[
=t_{A_{2n-2}} l_{A_1} l_{A_{2n-2}} \tilde{\tau}^{-1} \tilde{\theta}^{-1} l_{C_1} l_{C_2} l_{\delta_1} l_{\delta_2} l_{[\Sigma^4_g]}
\]

\[
=t_{A_{2n-2}} l_{A_1} l_{A_{2n-2}} (l_{A_{2n-3}} \cdots l_{A_1}) \tilde{\tau}^{-1} \tilde{\theta}^{-1} l_{\delta_1} l_{\delta_2} l_{[\Sigma^4_g]}
\]

\[
=t_{A_{2n-2}} l_{A_1} l_{A_{2n-3}} (l_{A_{2n-2}} \cdots l_{A_1}) (l_{A_{2n-3}} \cdots l_{A_1}) \tilde{\tau}^{-1} \tilde{\theta}^{-1} l_{\delta_1} l_{\delta_2} l_{[\Sigma^4_g]}
\]
Figure 7. Paths in $\Sigma^4_{2g}$.

\[ = t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-4}} (t_{A_{2n-2}} \cdots t_{A_1})^2 (t_{A_{2n-3}} \cdots t_{A_1})^2 \tilde{\tau}^{-1} = \cdots \]

\[ = (t_{A_{2n-2}} \cdots t_{A_1})^{2n-1} \tilde{\tau}^{-1} \tilde{\theta}^{-1} t_{\delta_1} t_{\delta_2} [\iota|\Sigma^4_{2g}] \]

It is easy to verify (using the Alexander method, for example) that the product $(t_{A_{2n-2}} \cdots t_{A_1})^{2n-1} \tilde{\theta}^{-1} [\iota|\Sigma^3_g]$ is equal to $\iota$ in $\text{Mod}(\Sigma^3_{g+n-1}; \{u_1, u_2\})$. Thus, we obtain:

\[ t_{A_{2n-2}} \cdots t_{A_1} t_{A_1} \cdots t_{A_{2n-2}} t_{B_0} \cdots t_{B_{2m+1}} = \tau^{-1} t_{\delta_1} t_{\delta_2} \]

This completes the proof of Proposition 5.1.

We take simple closed curves $c_1, c_2, c_3$ in $\Sigma^2_{2m+1}$ and points $u_1, \ldots, u_{2m}$ on the boundary of $\Sigma^2_{2m+1}$ as described in Figure 8. Let $\tau_i$ be a radial arc between $u_i$ and $u_{m+i}$. We denote by $\lambda \in \text{Mod}_{\partial \Sigma^2_{2m+1}}(\Sigma^2_{2m+1}; \{u_1, \ldots, u_{2m}\})$ a mapping class represented by a diffeomorphism which is positive 180-degree rotation inside of $\nu c_3$ and preserves the outside of $\nu c_3$, where $\partial \Sigma^2_{0}$ is the outermost boundary component of $\Sigma^2_{2m+1}$ in Figure 8.

**Proposition 5.2.** The following equality holds in $\text{Mod}_{\partial \Sigma^2_{0}}(\Sigma^2_{2m+1}; \{u_1, \ldots, u_{2m}\})$:

\[ t_{c_1} t_{c_2} \lambda^{-1} = t_{\delta_1} \cdots t_{\delta_{2m}} \tilde{\tau}_1^{-1} \cdots \tilde{\tau}_m^{-1}, \]
where \( \tilde{\gamma} \in \text{Mod}_{\partial \Sigma_0}(\Sigma_0^{2m+1}; \{u_1, \ldots, u_{2m}\}) \) is a lift of a half twist along \( \tau_i \) as described in Figure 1.

**Proof.** Denote by \( \tilde{\lambda} \) the involution of \( \Sigma_0^{2m+1} \) given by the 180-rotation. We define \( C(\Sigma_0^{2m+1}; \tilde{\lambda}) \) as follows:

\[
C(\Sigma_0^{2m+1}; \tilde{\lambda}) = \{ \varphi \in \text{Diff}^+_{\partial \Sigma_0}(\Sigma_0^{2m+1}; \{u_1, \ldots, u_{2m}\}) \mid \tilde{\lambda} \circ \varphi = \varphi \circ \tilde{\lambda} \}.
\]

We regard \( \tilde{\gamma} \) and \( t_{\tilde{\delta}} t_{\delta_{2m+i}} \) as elements in \( \pi_0(C(\Sigma_0^{2m+1}; \tilde{\lambda}), \text{id}) \). The quotient map \( \tilde{\lambda} : \Sigma_0^{2m+1} \to \Sigma_0^{2m+1}/\lambda \cong \Sigma_0^{2m+1} \) induces the following homomorphism:

\[
\tilde{\lambda} : \pi_0(C(\Sigma_0^{2m+1}; \tilde{\lambda}), \text{id}) \to \text{Mod}_{\partial \Sigma_0}(\Sigma_0^{m+1}; \{u_0\}, \{u'_1, \ldots, u'_{m}\}),
\]

where \( u_0 \in \Sigma_0^{m+1} \) is the image of the origin of the disk under \( \tilde{\lambda} \) and \( u'_i = /\lambda(u_i) \). The map \( \tilde{\lambda} \) is an isomorphism and the image \( \tilde{\lambda} \circ (\tilde{\gamma}^{-1} t_{\delta} t_{\delta_{2m+i}}) \) is a pushing map along some loop based at \( u_0 \). We can easily obtain the equality in Proposition 5.2 using these fact together with some equality in \( \pi_1(\Sigma_0^{m+1}\setminus \{u'_1, \ldots, u'_{m}\}, u_0) \). The details are left to the readers. \( \square \)

We remove \( m \) disks from the disk \( \Sigma_0 \) to obtain \( \Sigma_0^{m+1} \subset \Sigma_0 \). We obtain the surface \( \Sigma_0^{2m} \subset \Sigma_0^{2m+1} \) by attaching two \( \Sigma_0^{m+1} \)s to \( \Sigma_0^{2m+1} \):

\[
\Sigma_0^{2m} = \Sigma_0^{2m+1} \cup_{\partial \Sigma_0^{2m+1}} \Sigma_0^{2m+1} = \Sigma_0^{2m+1} \cup_{\partial \Sigma_0^{2m+1}} \Sigma_0^{2m+1}.
\]

Combining the equalities in Propositions 5.1 and 5.2, we obtain the following equality in \( \text{Mod}(\Sigma_0^{2m}; \{u_1, \ldots, u_{2m}\}) \):

\[
\eta_{n,g} \Phi_K(\eta_{n,g}) = \tau_{m}^{-4} \cdots \tau_{1}^{-4} \tau_{1} \cdots \tau_{m}^{-4}.
\]

Eventually, for arbitrarily large \( m \), we can find \( m \) disjoint bisections in the Lefschetz fibration \( f_{n,k} : E(n)_K \to S^2 \) each of which has self-intersection 0. Furthermore, each of the bisections has 4 branched points. Thus, all the bisections are tori.

**Remark 5.3.** It is in fact possible to generalize these examples to cover knot surgered elliptic surfaces which are not symplectic, when the knots used in the construction are not fibered. In this case, following the arguments in [3], we instead obtain a broken Lefschetz fibration on each knot surgered 4-manifold, where Seiberg-Witten basic classes still appear as a collection of torus bisections.

### 5.2. A counter-example to Stipsicz’s conjecture on fiber sum indecomposable Lefschetz fibrations

In [19] Stipsicz, and in [18] Smith, independently proved that if a Lefschetz fibration over the 2-sphere admits a section of self-intersection \(-1\), then it cannot be decomposed as a fiber sum of two non-trivial Lefschetz fibrations. Conversely, Usher proved that fiber sum of two non-trivial relatively minimal Lefschetz fibrations (i.e., without null-homotopic Lefschetz vanishing cycles), except for very special cases of fibrations on ruled surfaces, the result is always a minimal symplectic 4-manifold [21]. This left open a very intriguing conjecture of Stipsicz, whether or not any Lefschetz fibration that does not decompose as a non-trivial fiber sum would admit a 1-section, an affirmative answer to which would suggest blow-ups of Lefschetz pencils as the elementary building blocks of Lefschetz fibrations through fiber sums. However, in [17] Sato disproved this conjecture by observing that a particular genus 2 Lefschetz fibration constructed by Auroux in [2] admits no
section of self-intersection $-1$, even though, the total space of it is non-minimal and thus this fibration is necessarily fiber sum indecomposable by Usher’s theorem. Sato’s proof hangs on the existence of a sphere bisection in Auroux’s fibration of self-intersection $-1$, which we will recapture in terms of mapping class group factorizations using our Theorem 3.5.

We first give a quick review of the Auroux fibration. We take non-separating simple closed curves $c_1, c_2, c_3, c_4, c_5$ and a separating simple closed curve $\sigma$ in $\Sigma_2$ as described in the upper part of Figure 9. We denote by $t, t_\sigma \in \Mod(\Sigma_2)$ the right-handed Dehn twists along $c_1$ and $\sigma$, respectively. Auroux [2] proved that the following relation holds:

$$t_3 T^2 t_\sigma = 1,$$

where $t_2 = t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_1$ and $T = t_3 t_2 t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_1 t_2 t_3 t_4$. This equation gives rise to a genus-2 Lefschetz fibration $f : X \to S^2$ with 29 Lefschetz singularities, 28 of which are irreducible and one of which is reducible.

We take two points $s_1, s_2 \in \Sigma_2$ as in Figure 9, a disk neighborhood $D_i$ of $s_i$ and a point $u_i \in \partial D_i$ (See the lower part of Figure 9). We fix an identification $\Sigma_2 \setminus (D_1 \cup D_2) \cong \Sigma_2^2$. Denote by $t_i \in \Mod(\Sigma_2^2, \{u_1, u_2\})$ the Dehn twist along $c_i$ described in the lower part of Figure 9 and by $t_\sigma \in \Mod(\Sigma_2^2, \{u_1, u_2\})$ a lift of the Dehn twist $t_\sigma$ described in Figure 3. We put $t_3 = t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 t_9 t_1$ and $\tilde{T} = \tilde{t}_1 \tilde{t}_2 \tilde{t}_3 \tilde{t}_4 \tilde{t}_5 \tilde{t}_6 \tilde{t}_7 \tilde{t}_8 \tilde{t}_9 \tilde{t}_1$.

**Proposition 5.4.** The following relation holds in $\Mod(\Sigma_2^2, \{u_1, u_2\})$:

$$\tilde{t}_1 \tilde{t}_2 \tilde{t}_3 \tilde{t}_4 t_\sigma = \tilde{t}_1^2 \tilde{t}_2^2,$$

where $\tilde{t}_1 \in \Mod(\Sigma_2^2, \{u_1, u_2\})$ is a lift of a half twist described in Figure 1 preserving the path $\tau_1$ in Figure 9 and $\delta_i \subset \Sigma_2^2$ is a simple closed curve parallel to $\partial D_i$.

**Proof.** The element $\tilde{t}_\sigma$ is equal to $\tilde{t}_\sigma = (\tilde{t}_2 \tilde{t}_1)^3 (\tilde{t}_5 \tilde{t}_4)^3 [\tau_2]_{\Sigma_2^2}$, where $[\tau_2]_{\Sigma_2^2}$ is the mapping class described in the left side of Figure 10. Thus, the product $\tilde{T}^2 \tilde{t}_\sigma$ is calculated as follows:

$$\tilde{T}^2 \tilde{t}_\sigma = \tilde{T} \tilde{T} \tilde{t}_2 \tilde{t}_1 \tilde{t}_3 \tilde{t}_4 \tilde{t}_5 \tilde{t}_6 \tilde{t}_7 \tilde{t}_8 \tilde{t}_9 \tilde{t}_1 (\tilde{t}_2 \tilde{t}_1)^3 (\tilde{t}_5 \tilde{t}_4)^3 [\tau_2]_{\Sigma_2^2}$$
\[\tilde{\tau}_1\tilde{t}_2\tilde{t}_\sigma = \tilde{t}_2\tilde{t}_\sigma t_\delta t_\delta[\tau_2|\Sigma_2]\]

where the last equality follows from the chain relation in mapping class groups (cf. [7, Proposition 4.12]). Since the element \(\tilde{t}_2\) is commute with the class \([\tau_2|\Sigma_2]\], \(\tilde{t}_2\) induces the isotopy class of a self-diffeomorphism on \(\Sigma_2^2/(\tau_2|\Sigma_2)\) which preserves the fixed points of \(\tau_2\) pointwise. In particular, the element \(\tilde{t}_2\) induces an element in the group \(\text{Mod}(\Sigma_2^2;q_1,\ldots,q_6)\). It is easy to see that the induced element in \(\text{Mod}(\Sigma_2^2;\{u_1,u_2\})\) is equal to the pushing map \(\text{Push}(\alpha)\) along the loop \(\alpha\) described in Figure 10. Thus, the following relation holds in \(\text{Mod}(\Sigma_2^2;\{u_1,u_2\})\): \(\tilde{t}_2 = \tilde{\tau}_1^{-1}t_\delta t_\delta[\tau_2|\Sigma_2]\).

The product \(\tilde{\tau}_1\tilde{t}_2\tilde{t}_\sigma\) is calculated as follows:

\[\tilde{\tau}_1\tilde{t}_2\tilde{t}_\sigma = \tilde{t}_2\tilde{t}_\sigma t_\delta t_\delta[\tau_2|\Sigma_2]\]

**Figure 10.** A loop \(\alpha\) in the surface \(\Sigma_2^2/(\tau_2|\Sigma_2)\).

\[\tilde{\tau}_1\tilde{t}_2\tilde{t}_\sigma = \tilde{t}_2\tilde{t}_\sigma t_\delta t_\delta[\tau_2|\Sigma_2]\]

This completes the proof of Proposition 5.4.

Hence, using Theorem 3.5, we can construct a bisection of the Lefschetz fibration \(f : X \to S^2\) with self-intersection \((-1)\). Since this bisection has two branched points, the Euler characteristic of the bisection is equal to 2, i.e. it is a sphere. However, as Sato observes, the collection of \(-1\)-spheres in \(X\) should intersect the fiber of \(f\) precisely at 2 points \([17]\), and therefore this fibration does not have any section of self-intersection \(-1\).

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Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-9305, USA

E-mail address: baykur@math.umass.edu

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: khayano@cr.math.sci.osaka-u.ac.jp