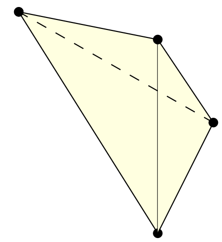
Alejandro Morales

I. Kostant partition function and flow polytopes

$K_{A_n}(a_1, \ldots, a_{n+1})$ number of ways of writing $(a_1, \ldots, a_{n+1})$ as an $\mathbb{N}$-combination of **positive roots** $e_i - e_j$, $i < j$.



$K_{A_3}(1, 0, 0, -1) = 4$:

$$\begin{aligned}(1,0,0,-1) &= e_1 - e_4\\ &= (e_1 - e_3) + (e_3 - e_4)\\ &= (e_1 - e_2) + (e_2 - e_4)\\ &= (e_1 - e_2) + (e_2 - e_3) + (e_3 - e_4)\end{aligned}$$

- Kostant (1958) used them to give formulas for weight multiplicities of irreducible representations of semisimple Lie algebras. Lusztig (1983) studied a q-analogue.

Combinatorial approach (Baldoni-Vergne 2001)

View ways counted in $K_{A_n}(\cdot)$ as **lattice points** of a **flow polytope** $\mathcal{F}_G(a_1, \ldots, a_{n+1})$

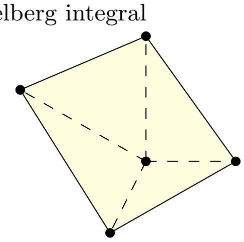
$P_n := \mathcal{F}_G(1, 0, \ldots, 0, -1)$ volume$(P_n) = K_G(1, 2, 3, \ldots) = C_1 C_2 \cdots C_{n-2}$ $C_i = \frac{1}{i+1}\binom{2i}{i}$

volume equals # lattice points similar to *permutahedra*

Zeilberger 1999 by identity related to Selberg integral

Results

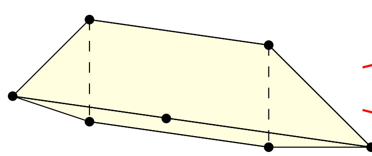
- [1] Extend from Lie type A to Lie types B, C, D



$K_{D_2}(2, 0, 0) = 5$

- [2][3] Connection to **space of diagonal harmonics** DH_n, **shuffle conjecture**
  new polytope $Q_n := \mathcal{F}_G(1, 1, \ldots, -n)$, $n!$ vertices, volume$(Q_n) = \frac{\binom{n}{2}!}{\prod_i (2i-1)^{n-i}} C_1 \cdots C_{n-1}$

$K_{A_3}(1, 1, 1, -3) = 7$:



→ q, t-Catalan numbers

→ Hilbert series space DH_n

- [4] $\mathcal{F}_G(1, 0, \ldots, -1)$ when G is planar $\mathcal{F}_G \equiv$ **order polytope** of poset from dual of G
  Corollary: certain Kostant partition functions count linear extensions of posets.

- [5] lattice/Ehrhart theory flow polytopes parallel to **generalized permutahedra**

References

[1] K. Mészáros and A.H. M., *Int. Math. Res. Not.*, rnt212:830–871, 2015.

[2] K. Mészáros, A.H. M., B. Rhoades. The polytope of Tesler matrices. *Selecta Math.* bf 23, 2017.

[3] R.I. Liu, K. Mészáros, and A.H. M. , `arXiv:1610.08370`.

[4] K. Mészáros, A.H. M., and J. Striker. , `arXiv:1510.03357`.

[5] K. Mészáros, A.H. M., Lidskii formulas for lattice points of flow polytopes, *in preparation*.

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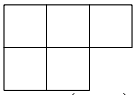
II. Hook formulas for skew shapes

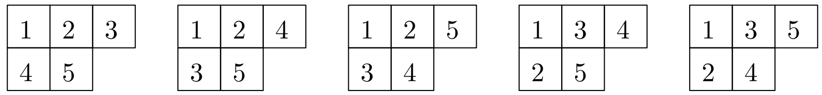
The irreducible representations of the symmetric group S_n are indexed by partitions λ of n.

The **dimension** f_λ of the irreducible representation counts **standard tableaux**: fillings of the diagram of λ with all entries $1, 2, \ldots, n$ increasing in rows and columns (Young 1900)



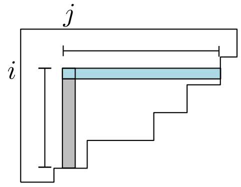
$\lambda = (3, 2)$

$f^{(3,2)} = 5$

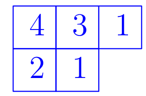


f^λ has a product formula: the **hook-length formula** (Frame-Robinson-Thrall 1954)

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} \text{hook}(i,j)}$$



$$f^{(3,2)} = \frac{5!}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = 5$$



hooks

Applications:

uniform sampling

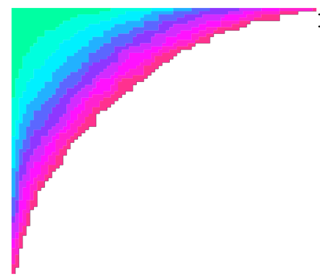
- Greene-Nijenhuis-Wilf 79
- Novelli-Pak-Stoyanovski 97

1	2	5	6	10	14	17	22	46	49
3	4	8	13	19	23	26	34	50	56
7	9	18	25	31	33	37	40	62	73
11	15	27	29	38	51	52	55	66	74
12	16	28	35	39	54	64	65	72	82
20	24	36	41	43	59	68	75	78	85
21	32	47	58	61	67	71	87	90	93
30	44	48	60	63	70	80	88	95	97
42	53	57	76	77	79	81	89	96	98
45	69	83	84	86	91	92	94	99	100

†

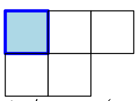
Limit shapes (Plancherel)

- $GL(\infty)$: Vershik-Kerov 77
- probability: Logan-Shepp 78
- free probability: Biane 98



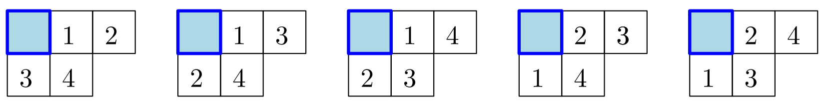
‡

The **dimension** of irreducible *calibrated* representations of the Hecke algebra counts **standard tableaux of skew shape**: fillings of λ/μ with all entries $1, 2, \ldots, n$ increasing in rows and columns (Ram 2004)

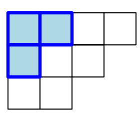


$\lambda/\mu = (3, 2)/(1)$

$f^{(3,2)/(1)} = 5$



$f^{\lambda/\mu}$ has **no product formula**



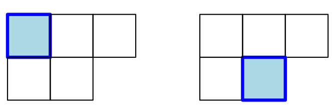
has 61 tableaux

prime

alternating permutations (André 1881)

Naruse announced formula for $f^{\lambda/\mu}$ as a positive sum of products using **equivariant Schubert calculus**.

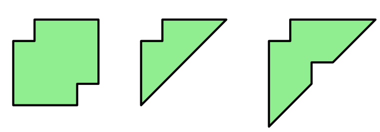
$$f^{\lambda/\mu} = n! \sum_{D} \prod_{(i,j)\in\overline{D}} \frac{1}{\text{hook}(i,j)}$$



$$f^{(3,2)/(1)} = 4! \left(\frac{1}{1{\cdot}1{\cdot}2{\cdot}3} + \frac{1}{1{\cdot}2{\cdot}3{\cdot}4} \right) = 5$$

Results

- two q-analogues of Naruse formula [1]
- elementary proof of Naruse formula using **Gessel-Viennot theory** [2]
- asymptotics for $f^{\lambda/\mu}$ [3]   **Goal:** find limit shapes of $f^{\lambda/\mu}$.
- new family of skew shapes with product formulas for $f^{\lambda/\mu}$ [4]



References

[1] A.H. Morales, I. Pak, and G. Panova. `arXiv:1512.08348`, [2] `arXiv:1610.07561` *accepted SIDMA*, [3] `arXiv:1610.07561`, [4] in preparation.

†, ‡ images made using from D. Romik *MacTableaux*.

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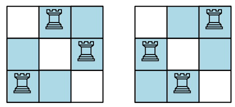
III. q-analogue of placements of non-attacking rooks

In **complexity theory** computing the **permanent** of a (0-1) matrix is hard (#P-**complete**) (Valiant 79)

$$\text{perm}(A) = \sum_{w\in S_n} A_{1w_1}\cdots A_{nw_n}$$

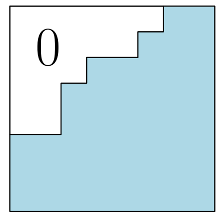
For an $n\times n$ 0-1 matrix A, perm(A) = # placements of n non-attacking rooks on **support** of A.

$$\text{perm}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} = 2$$



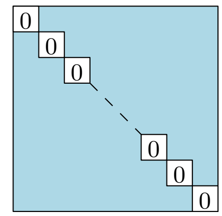
some nice cases:

diagram of partition



$\text{perm}(A) = \prod_i(\lambda_i - i + 1)$

derangements



$d_n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)!$

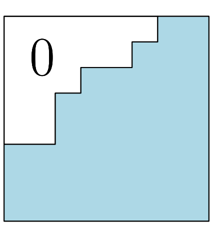
Results

new q-analogue of rook placements [1]: for $S \subset \{1,2,\ldots,n\}\times\{1,2,\ldots,n\}$

$M_q(S) :=$ #{invertible matrices A entries in finite field $\mathbb{F}_q$ support in S} $\subseteq \text{GL}_n(\mathbb{F}_q)$

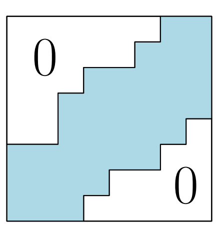
- **Theorem:** enumerative q-analogue of rook placements:
  i.e. $\lim_{q\to 1} M_q(S) = \#\{\text{rook placements in } S\}$
- not always polynomial in q, can be **P**olynomial **O**n **R**esidue **C**lasses (Stembridge 98)
- polynomial in q when support is diagram of partition (Haglund 97) and when support is a skew diagram [2]
- **Theorem:** polynomial in q when **zeros** are on **inversions** of any permutation [4] proved using **coding theory** (settling conjecture from [2])
- not known if there are non **PORC** examples

diagram partition



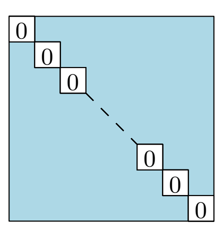
polynomial in $\mathbb{N}[q]$
(Hagllund)

diagram skew partition



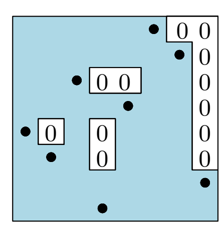
polynomial in $\mathbb{N}[q]$
[2]

diagonal



polynomial in $\mathbb{Z}[q]$
[1]

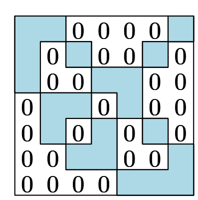
diagrams of permutations



$w = 67351284$

polynomial in $\mathbb{Z}[q]$
(conjecture [2])
(proved [3][4])

Fano plane



not polynomial in q
(Stembridge)

References

[1] J. B. Lewis, R. Liu, A.H. Morales, G. Panova, Sam S. V, and Y. X. Zhang. *J. Comb.*, 2(3):355–395, 2011.

[2] A. Klein, J.B. Lewis, and A.H. Morales. *J. Alg. Comb.*, 39(2):429–456, 2014.

[3] J.B. Lewis and A.H. Morales. *J. of Combin. Theory Ser. A*, 137:273–306, 2016.

[4] J.B. Lewis and A.H. Morales. 2017, in preparation.

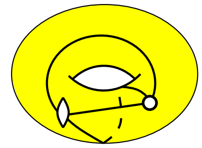
IV. Colored factorizations in S_n and $\mathrm{GL}_n(\mathbb{F}_q)$

Structure constants of the group algebra of the symmetric group count **factorizations**

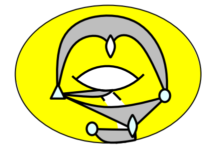
$$F_{\lambda,\mu} = \#\{(\pi_1,\pi_2) \mid \pi_1,\pi_2 \in S_n,\ \text{ cycle type } \pi_1 = \lambda,\ \text{cycle type } \pi_2 = \mu,\quad \pi_1\pi_2 = (1\,2\ldots n)\}$$

$F_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}}$ Case of k permutations:

related to **maps**, **constellations**, **matrix integrals**, and **Hurwitz problem** of counting ramified covers of the sphere.



map



constellation

Example

- if $\lambda^{(i)} = 21^{n-2}$, $k = n-1$ $\quad F_{21^{n-2},\ldots,21^{n-2}} = n^{n-2}$.
  factorizations $(1,2,\ldots,n)$ into $n-1$ *transpositions* (Hurwitz 1891, Dénes 1959)

Positive formulas for general $F_{\lambda,\mu}$ have **exponentially many terms** (Goupil-Schaeffer 1998)

Approach: (Harer-Zagier 86, Jackson 88, Schaeffer-Vassilieva 08)

In the *generating function* of $F_{\lambda/\mu}$ do a **change of basis** like $x^r \to x(x-1)\cdots(x-r+1)$ and obtain new coefficients $C_{\alpha,\beta}$.

$C_{\alpha,\mu}$ count factorizations with **colored** cycles, usually they have **nicer formulas**

Results (symmetric group S_n)

- two factors [1]

$F_{\lambda,\mu}$ → change of basis → $C_{\alpha,\beta} = \dfrac{n(n-\ell(\alpha))!(n-\ell(\beta))!}{(n+1-\ell(\alpha)-\ell(\beta))!}$

- k factors (Jackson 88, bijective proof [2])

$F_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ → change of basis → $C_{\alpha^{(1)},\ldots,\alpha^{(k)}} = n!^{k-1}\cdot S^{n-1}_{\ell_1-1,\ldots,\ell_k-1} / \prod_{i=1}^k \binom{n-1}{\ell_i-1}$.

where $\ell_i = \ell(\alpha^{(i)})$, $\quad S^n_{r_1,\ldots,r_k} := \#\left\{(S_1,\ldots,S_n)\ ,\mid\ \ S_i \subsetneq [k], r_j \text{ sets } S_i \text{ contain } j\right\}$,

---

In $\mathrm{GL}_n(\mathbb{F}_q)$ (Lewis-Reiner-Stanton 2013; Huang-Lewis-Reiner 2015):

- analogue of **long cycle** $(1,2,\ldots,n)$ → **Singer cycle**
- analogue of **number of cycles** of π → **fixed space dimension** of matrix
- number of factorizations Singer cycle in $\mathrm{GL}_n(\mathbb{F}_q)$ into n *reflections* is $(q^n-1)^n$

Results [3]

$F_{r,s}(n,q) = \#\{(A,B) \mid A,B \in \mathrm{GL}_n(\mathbb{F}_q),\quad A\cdot B = \text{fixed Singer cycle},\quad \text{dimension fixed space } A,B = r,s\}$

$F_{r,s}(n,q)$ → change of basis → $C_{r,s}(n,q) = \dfrac{q^{rs-r-s}(q^n-q^r-q^s+1)}{(q-1)[n-1]!}\dfrac{[n-r-1]![n-s-1]!}{[n-r-s]!}$

References

[1] A.H. Morales and E.A. Vassilieva. *Electron. J. of Combin.*, 20(2), 2013.

[2] O. Bernardi and A.H. Morales. *Adv. in Appl. Math.*, 50(5):702–722, 2013.

[3] J.B. Lewis and A.H. Morales. *Euro. J. Comb.*, 58:75–95, 2016.