M624 HOMEWORK – SPRING 2024

Prof. Andrea R. Nahmod

<u>SETS 1 & 2 - Due 02/27/2024</u>

From Chapter 3 (pp 145-146 -Section 5): 14, 16, 19, 23, 32 (some of you did part 16a) in M623.Please redo and complete the problem).

From Chapter 3 (pp 153): 4.

<u>Additional Questions (Chapter 3)</u>: After you had read carefully –as assigned in class– the proofs of Lemma 3.'3 and Theorem 3.14 do the following

1) Explain why $J_F(y) - J_F(x) \leq \sum_{n: x < x_n \leq y} \alpha_n \leq F(y) - F(x)$ (proof of Lemma 3.13).

2) Show rigorously that $J_F(x) - F(x)$ is continuous (in proof of Lemma 3.13).

3) Rewrite explaining fully the proof of Theorem 3.14 in Chapter 3. Note you need to solve and use exercise 14 (given above in Chapter 3).

The following problems concern Chapter 3 Section 2 of SS III. Read it and read the 2 Handouts I posted. Then do:

From Chapter 3 (pp 145-146 -Section 5): 1 [part c) is involved; see handout]; 2.

<u>SET 3 - Due 03/05/2024</u>

From Chapter 4 : Recall carefully the proof of both Theorem 2.2 (Riesz-Fisher) on Chaper 2 (p. 70) and then the one for Theorem 1.2 Chapter 4 (p. 159)

From Chapter 4 (pp 193-194): 1, $2^{\dagger}$, 3, 4 (show completeness only), 5, 6a), 7, 8a), 10.

† to show that $f - g$ is *orthogonal* to g you need to show that $\langle f - g, g \rangle = 0$.

Pb.I. Consider $f \in L^2([-\pi, \pi])$ and <u>assume</u> that $\sum_{n \in \mathbb{Z}} a_n e^{inx} = f(x)$ a.e. x. Show that on any subinterval $[a, b] \subset [-\pi, \pi]$,

$$\int_a^b f(x)\,dx = \sum_n \int_a^b a_n\, e^{inx}\,dx.$$

In particular if $g(x) = \int_a^x f(y)dy$, the Fourier coefficients and series of $g(x)$ can be obtained from a_n, the Fourier coefficients of f.

Pb. II. For $0 < \alpha < 1$, we say that a function f is C^α-Hölder continuous with exponent α if there exists a constant $c = c_\alpha > 0$ such that $|f(x) - f(y)| \leq c\,|x - y|^\alpha$ for all x, y.

For $k \in \mathbb{N}$, we can also define the space $C^{k,\alpha}$ to be that of functions which are k-th times differentiable and whose k-th derivative is C^α-Hölder continuous (we could relabel C^α as $C^{0,\alpha}$). Consider now f a 2π-periodic $C^{k,\alpha}$ function. If a_n are the Fourier coefficients of f, show that for some $C > 0$ independent of n,

$$|a_n| \leq \frac{C}{|n|^{k+\alpha}}$$

Bonus Problem: 2*a)b) from Chapter 4, pp 202.

SET 4 - Due 03/12/2024

From Chapter 4 (pp 195-197): 11, 12, 13, 18, 19, 20

Pb. I. Consider the subspace $\mathcal{S}$ of $L^2([0,1])$ spanned by the functions: 1, x, and x^3.
a) Find an orthonormal basis of $\mathcal{S}$.
b) Let $P_{\mathcal{S}}$ denote the orthogonal projection on the subspace $\mathcal{S}$, compute $P_{\mathcal{S}} x^2$.

Pb. II. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a periodic function with period p; that is $\phi(x+p) = \phi(x)$, $\forall x \in \mathbb{R}$. Assume that ϕ is integrable on any finite interval.

(a) Prove that for any $a, b \in \mathbb{R}$

$$\int_a^b \phi(x)dx = \int_{a+p}^{b+p} \phi(x)dx = \int_{a-p}^{b-p} \phi(x)dx$$

(b) Prove that for any $a \in \mathbb{R}$

$$\int_{-p/2}^{p/2} \phi(x+a)dx = \int_{-p/2}^{p/2} \phi(x)dx = \int_{-p/2+a}^{p/2+a} \phi(x)dx$$

In particular we have that $\int_a^{a+p} \phi(x)\,dx$ does not depend on a, as we discussed in class.

SET 5 - Due 03/28/2024

Assigned Reading (in class) from Chapter 4 of [Stein-Shakarchi Vol 3]:
Remarks (a), (b), (c) on pages 184-185.

Turn in:

From Chapter 4 (pp 187): Read and rewrite filling in all details the Proof of Proposition 5.5.

From Chapter 4 (pp 189): Read and rewrite filling in all details the Proof of Proposition 6.1.

From Chapter 4 (pp 197-202): 21, 22, 23, 25, 26, 28.

SET 6 - DUE 04/04/2024

From Chapter 4 (pp 197-202): 30, 32, 33.

Bonus Problems : 29* (p 199-200) and 6* (p. 203-204). These are about Fredholm's Alternative for compact operators.

SET 6 - DUE 04/18/2024

From Chapter 5 (pp 253-255): 1, 9 (see definition and example below).

Definition: A Fourier multiplier operator T on $\mathbb{R}^d$ is a linear operator on $L^2(\mathbb{R}^d)$ determined by a bounded function m (the multiplier) such that T is defined by the formula

$$\widehat{T(f)}(\xi) := m(\xi)\widehat{f}(\xi)$$

for all $\xi \in \mathbb{R}^d$ and any $f \in L^2(\mathbb{R}^d)$.

Examples. The bounded linear operator $P_N : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $\widehat{P_N(f)}(\xi) := \chi_{[-N,N]}(\xi)\,\widehat{f}(\xi)$ is one such operator. In fact is an orthogonal projection.

Another well known one is the *Hilbert Transform* $\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $\widehat{\mathcal{H}(f)}(\xi) := -isgn(\xi)\,\widehat{f}(\xi)$. The operator $\mathcal{H}$ is bounded and linear on L^2. That is bounded foillows from the fact that $-isgn(\xi)$ is bounded point-wise by 1 and Theorem 1.1 that says the Fourier transform is unitary on $L^2(\mathbb{R}^d)$

From Chapter 5 (pp 260-261): 6.

Additional Problem Carefully read and rewrite on your own (justifying and filling the gaps as necessary) Lemma 1.2 on page 209 proving that $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$

From Chapter 6: Read/Study the proofs in Section 1.

From Chapter 6 (pp 312 313): 1 (change $\mathcal{M}$ to be a non-empty algebra), 2a), 8.

Extra Problem (do not to turn in): **From Chapter 5 (pp 260-261)**: 5

SET 7 - DUE 04/25/2024

From Chapter 6 (pp 317-322): 5, 10, 11a)b), 16a)b)

Additional Problems:

(A1) Let ν be a finite signed measure on $(X, \mathcal{M})$. Show that for any $E \in \mathcal{M}$

$$|\nu|(E) =$$

$$= \sup\{\sum_{k=1}^{K} |\nu(E_k)| \, : \, E_1, \ldots E_K \text{ are disjoint and } E = \cup_{k=1}^{K} E_k \} \tag{1}$$

$$= \sup\{\sum_{k=1}^{\infty} |\nu(E_k)| \, : \, E_1, E_2, \ldots \text{ are disjoint and } E = \cup_{k=1}^{\infty} E_k \} \tag{2}$$

$$= \sup\{|\int_E f d\nu| \, : \, |f| \leq 1\} \tag{3}$$

You may want to proceed for example by proving that (1) $\leq$ (2) $\leq$ (3) $\leq$ (1).

(A2) Let $F \in BV([a,b])$ and right continuous. Let $G(x) = |\mu_F|([a,x])$. Show that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$. To do so you may proceed by proving:

1) $T_F \leq G$ (use definition of T_F).

2) $|\mu_F(E)| \leq \mu_{T_F}(E)$ for any Borel set E (do for an interval first).

3) Show that $|\mu_F| \leq \mu_{T_F}$ and hence $G \leq T_F$ (use (A1)).

Do (but do not turn in): 9, 16c)d)e)f).