Math 623: notes on product sets, measures, etc. (end of Section 3, Ch. 2 Stein-Shakarchi III).

We stated

Recall [redacted] in class the following:

1) **Corollary 3.3** (Corol. to Tonelli): If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (w.r.t m Lebesgue msr of $\mathbb{R}^d$ $d = d_1 + d_2$) then for a.e $y \in \mathbb{R}^{d_2}$ the slice $E^y := \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$ is a measurable set of $\mathbb{R}^{d_1}$.

Moreover $m(E^y)$ $\left(= \int_{\mathbb{R}^{d_1}} \chi_{E^y}(x)\, dm(x)\right)$ a function of y is measurable (w.r.t to $m_{\mathbb{R}^{d_2}}$) and

$$\int_{\mathbb{R}^{d_2}} m(E^y)\, dm_{\mathbb{R}^{d_2}}(y) = m(E)$$

We did not have time to prove Corollary 3.3. So FIRST read its proof from the book and then continue with these notes for the rest to end with Chapter 2.

2) **Definition**: For $E_1 \subseteq \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$ sets, the set $E := E_1 \times E_2 \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} (\simeq \mathbb{R}^d)$ is called a **product set**

3) **Proposition 3.4**: If $E = E_1 \times E_2$ is a measurable set of $\mathbb{R}^d$ (ie. w.r.t $m_{\mathbb{R}^d}$) and (w.r.t $\mathbb{R}^{d_2}$) $m_*(E_2) > 0$, then E_1 is measurable (w.r.t $m_{\mathbb{R}^{d_1}}$)

The converse of Prop. 3.4 reads as follows:

<u>Proposition 3.6</u>: Suppose E_1, E_2 are measurable sets of $\mathbb{R}^{d_1}$ and of $\mathbb{R}^{d_2}$ respectively. Then the product set $E := E_1 \times E_2$ is a measurable set of $\mathbb{R}^d$

Moreover: $\underset{\mathbb{R}^d}{m}(E) = \underset{\mathbb{R}^{d_1}}{m}(E_1)\, \underset{\mathbb{R}^{d_2}}{m}(E_2)$.

If one of E_1 and/or E_2 has measure 0 then $m(E)=0$

<u>To prove</u> Proposition 3.6 we need the following:

<u>Auxiliary Lemma 3.5</u>: If $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$ then

$$m_*(E_1 \times E_2) \leq m_*(E_1) \cdot m_*(E_2)$$

(l.h.s is outer msr in $\mathbb{R}^d$; r.h.s. one is outer in $\mathbb{R}^{d_1}$ the other outer in $\mathbb{R}^{d_2}$)

<u>Assuming the Auxiliary Lemma 3.5 let's prove</u>

<u>Prop. 3.6</u>:

WTS that E is measurable: since E_1, E_2 are measurable, $\exists\, G_1 \subset \mathbb{R}^{d_1}$ G_δ set $G_1 \supset E_1$

and $G_2 \subseteq \mathbb{R}^{d_2}$, G_δ set $G_2 \supset E_2$ such that

$$m_{\mathbb{R}^{d_1}}(G_1 - E_1) = 0 = m_{\mathbb{R}^{d_2}}(G_2 - E_2).$$

WHY? (prove) [Now $G_1 \times G_2$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \equiv \mathbb{R}^d$ and $G := \underline{G_1 \times G_2}$ is a G_δ set in $\mathbb{R}^d$ s.t.

$$\underbrace{(G_1 \times G_2)}_{=:G} \setminus \underbrace{(E_1 \times E_2)}_{=:E} \subseteq [(G_1 \setminus E_1) \times G_2] \cup [G_1 \times (G_2 \setminus E_2)]$$

By the Auxil Lemma 3.5 we can then conclude that $m_*(G \setminus E) = 0 \Rightarrow E$ is measurable.

- The fact that $m(E) = m(E_1)\, m(E_2)$ now follows from Corol 3.3 (to Tonelli).

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<u>Proof of Auxiliary Lemma 3.5:</u> Let $\varepsilon > 0$ be given. Let $\{Q_k^{(1)}\}_{k \geq 1}$ be cubes in $\mathbb{R}^{d_1}$ and $\{Q_j^{(2)}\}_{j \geq 1}$ be cubes in $\mathbb{R}^{d_2}$ such that:

(i) $E_1 \subseteq \bigcup_{k=1}^{\infty} Q_k^{(1)}$; $E_2 \subseteq \bigcup_{j=1}^{\infty} Q_j^{(2)}$

(ii) $\sum_{k=1}^{\infty} |Q_k^{(1)}| \leq m_*(E_1) + \varepsilon$, $\sum_{j=1}^{\infty} |Q_j^{(2)}| \leq m_*(E_2) + \varepsilon$

(where $m_*(E_1)$ is outer on $\mathbb{R}^{d_1}$ $m_*(E_2)$ is outer on $\mathbb{R}^{d_2}$)

Since $E_1 \times E_2 \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_k^{(1)} \times Q_j^{(2)}$

$$\Longrightarrow \quad m_*(E_1 \times E_2) \leq \sum_{j,k=1}^{\infty} |Q_k^{(1)} \times Q_j^{(2)}|$$

by the subadditivity of outer measure. But

$$\sum_{j,k=1}^{\infty} |Q_k^{(1)} \times Q_j^{(2)}| \underset{\downarrow}{=} \left(\sum_{k=1}^{\infty} |Q_k^{(1)}|\right)\left(\sum_{j=1}^{\infty} |Q_j^{(2)}|\right)$$

double sum, indep indexes

$$\leq \left(m_*(E_1)+\varepsilon\right)\left(m_*(E_2)+\varepsilon\right)$$

$$\leq m_*(E_1)\, m_*(E_2) + C\cdot\varepsilon$$

~~for~~ (for C fixed $C>0$) provided $m_*(E_1) \neq 0 \neq m_*(E_2)$

Since $\varepsilon > 0$ is arbitrary we then get (by $\varepsilon \to 0$) that

$$m_*(E_1 \times E_2) \leq m_*(E_1)\, m_*(E_2).$$

[Want to avoid having $0.\infty$]

If - say - $m_*(E_1) = 0$ then consider the sequence of sets $E_2 \cap B(0,j) =: E_j^{(2)}$ where $B(0,j)$ = ball centered at 0 and of radius $j \geq 1$ on $\mathbb{R}^{d_2}$

Then $E_j^{(2)} \underset{j\to\infty}{\nearrow} E_2$ and $E_1 \times E_j^{(2)} \underset{j\to\infty}{\nearrow} E_1 \times E_2$

By repeating the first part of the proof, we have that

$m_*(E_1 \times E_j^{(2)}) = 0$, hence $m_*(E_1 \times E_2) = 0$

Corollary 3.7: Suppose that f is a measurable function on $\mathbb{R}^{d_1}$. Let $\tilde{f} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be defined as $\tilde{f}(x,y) := f(x)$.

Then $\tilde{f}$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (w.r.t $m_{\mathbb{R}^d}$)

$d = d_1 + d_2$.

Proof: Let $a \in \mathbb{R}$ and $E_1 := \{x \in \mathbb{R}^{d_1} : f(x) < a\}$

Since f is measurable on $\mathbb{R}^{d_1} \Rightarrow E_1$ is measurable

$\Rightarrow E_1 \times \mathbb{R}^d = \{(x,y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : \tilde{f}(x,y) < a\}$

is measurable w.r.t $m_{\mathbb{R}^d}$ by Prop. 3.6.

Thus $\tilde{f}$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ by definition.

Corollary 3.8: Suppose $f(x)$ is a non-negative function on $\mathbb{R}^d$ and let

$$A := \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \le y \le f(x)\}$$

Then: (i) f is measurable on $\mathbb{R}^d \Leftrightarrow A$ is measurable on $\mathbb{R}^{d+1}$

(ii) If (i) holds $\Rightarrow m(A) = \int_{\mathbb{R}^d} f(x)\,dm(x)$.

Proof (i): Define $F(x,y) := y - f(x)$ and note that by Corol 3.7, F is measurable on $\mathbb{R}^{d+1}$. But then A is a measurable set on $\mathbb{R}^{d+1}$ since it can be realized as the intersection of

$\{(x,y) \in \mathbb{R}^d \times \mathbb{R} \ / \ y \geq 0\}$ and
$\{(x,y) \in \mathbb{R}^d \times \mathbb{R} \ / \ F(x,y) \leq 0\}$

Now, for the converse suppose A is measurable then for each $x \in \mathbb{R}^{d_1}$;

(slice) $A_x := \{ y \in \mathbb{R} : (x,y) \in A \}$ is a closed segment $[0, f(x)]$.

By Corol 3.3 $(x \leftrightarrow y)$ $m(A_x)$ is a measurable function on $\mathbb{R}^{d_1}$. But $m(A_x) = f(x) \Rightarrow$ f is a measurable function on $\mathbb{R}^{d_1}$.

Furthermore,

$$m(A) = \int_{\mathbb{R}^{d+1}} \chi_A(x,y) \, dm_{\mathbb{R}^{d+1}}$$

$$= \int_{\mathbb{R}^{d_1}} m(A_x) \, dm(x) = \int_{\mathbb{R}^{d_1}} f(x) \, dm(x)$$

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<u>Proposition 3.9</u> : If f is a measurable function on $\mathbb{R}^d$, then $\tilde{f}(x,y) := f(x-y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$

<u>Proof:</u> Define $E := \{ w \in \mathbb{R}^d \,/\, f(w) < a \}$

WTS that <u>if</u> E is measurable subset of $\mathbb{R}^d \Rightarrow$ the set $\tilde{E} := \{ (x,y) : x-y \in E \}$ is a measurable subset of $\mathbb{R}^d \times \mathbb{R}^d$.

First note that if $\mathcal{O}$ is open in $\mathbb{R}^d \rightarrow \tilde{\mathcal{O}}$ is open in $\mathbb{R}^d \times \mathbb{R}^d$ and that if G is a G_δ set in $\mathbb{R}^d \rightarrow \tilde{G}$ is a G_δ set in $\mathbb{R}^d \times \mathbb{R}^d$.

Recall now that any measurable set can be written as the difference of a G_δ set and a set of measure zero.

So consider $Z \,/\, m(Z) = 0$ <u>WTS</u> that then $m(\tilde{Z}) = 0$: Consider $\mathcal{O}$ open in $\mathbb{R}^d$ and let $\tilde{\mathcal{O}}_R := \tilde{\mathcal{O}} \cap B(0,R)$ where $B(0,R)$ is the <u>ball</u> <u>on</u> <u>$\mathbb{R}^d$</u> centered at 0 of radius R

Then:

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$$\chi_{\tilde{\mathcal{O}}_k}(x,y) = \chi_{\mathcal{O}}(x-y)\,\chi_{B(0,k)}(y) \Rightarrow$$

$$m(\tilde{\mathcal{O}}_k) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_{\mathcal{O}}(x-y)\,\chi_{B(0,k)}(y)\, dm$$

$$\underset{\text{Tonelli}}{=} \int_{\mathbb{R}^d} \Big[\underbrace{\int_{\mathbb{R}^d} \chi_{\mathcal{O}}(x-y)\, dm(x)} \Big] \chi_{B(0,k)}(y)\, dm(y)$$

$$= m(\mathcal{O})\, m(B(0,k))$$ (each of these m are on $\mathbb{R}^d$)

Transl. Invariance of Lebesgue msr

Now if $Z \subseteq \mathbb{R}^d$ is / $m(Z) = 0 \Rightarrow$

$\exists\ \mathcal{O}_n \subset \mathbb{R}^d$, $Z \subset \mathcal{O}_n$ / $m(\mathcal{O}_n) \to 0$

as $n \to \infty$. Then $\tilde{Z} \cap B(0,k) \subset \tilde{\mathcal{O}}_n \cap B(0,k)$

and $m(\tilde{\mathcal{O}}_n \cap B(0,k)) \longrightarrow 0$ as $n \to \infty$

for each fixed $k \geq 1$. Hence $m(\tilde{Z} \cap B(0,k)) = 0$

for all k $\Rightarrow m(\tilde{Z}) = 0$ as desired

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