Math 545 Solution of Midterm 2 Spring 2007

1. (17 points) Let θ be an angle, such that $\sin(\theta) \neq 0$, and let $A := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ be the matrix of the rotation of $\mathbb{R}^2$ about the origin by angle θ counterclockwise.

   (a) The characteristic polynomial of A is $h(x) = x^2 - 2\cos(\theta) + 1$.

   (b) The minimal polynomial $m(x)$ of A is equal to its characteristic polynomial $x^2 - 2\cos(\theta) + 1$. Two ways to see it are: If we work over the complex numbers, then the equality $h(x) = m(x)$ follows, since the two roots $\cos(\theta) + \sin(\theta)i$ and $\cos(\theta) - \sin(\theta)i$ of $h(x)$ are distinct, and $m(x)$ and $h(x)$ have the same set of roots, and $m(x)$ divides $h(x)$, by the Cayley Hamilton Theorem. Over the real numbers the equality $m(x) = h(x)$ follows, since $h(x)$ is a prime polynomial, $m(x)$ has positive degree, and $m(x)$ divides $h(x)$.

   (c) A is similar to a diagonal matrix in $M_2(\mathbb{C})$, since the minimal polynomial factors as a product of linear terms with distinct roots
   $m(x) = (x - [\cos(\theta) + \sin(\theta)i])(x - [\cos(\theta) - \sin(\theta)i])$.

   (d) A is not similar to a diagonal matrix in $M_2(\mathbb{R})$, since its minimal polynomial is prime in $\mathbb{R}[x]$.

2. (17 points) Set $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

   (a) The characteristic polynomial of A is $x^2 + 1$. Its two eigenvalues are i and $-i$.

   (b) A basis of $\mathbb{C}^2$ consisting of eigenvectors of A. We find first a basis for the i-eigenspace $null(A - iI)$, then for the $-i$ eigenspace $null(A + iI)$, and take their union as a basis for $\mathbb{C}^2$.
   $A - iI = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$.
   Thus, $null(A - iI)$ is spanned by $v_1 := \begin{pmatrix} i \\ 1 \end{pmatrix}$.
   $A + iI = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$.
   Thus, $null(A + iI)$ is spanned by $v_2 := \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

   (c) Find an invertible matrix P and a diagonal matrix D, both in $M_2(\mathbb{C})$, such that $P^{-1}AP = D$.

Answer: Take $P := (v_1 v_2) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

3. (17 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by multiplication by $A = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$.

   (a) The characteristic polynomial of T is $h(x) = (x - 1)^2$.

   (b) The minimal polynomial $m(x)$ of T divides $h(x)$. If $m(x) = x - 1$, then $A = I$, which is false. Hence, $m(x) = (x - 1)^2$.

(c) T is not diagonalizable, since its minimal polynomial is a product of linear terms with repeated roots.

(d) The unique eigenvalue of T is the scalar 1.

(e) As a basis for the 1-eigenspace of T we can take $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(f) Find an upper triangular matrix B and an invertible matrix P, such that $B = P^{-1}AP$. Carefully explain, in complete sentences, your method for finding P. Credit will not be given for an answer obtained by trial and error.

Answer: The proof of the Triangular Form Theorem dictates, that we should choose a basis $\{v_1, v_2\}$ for $\mathbb{R}^2 = null((A-I)^2)$, so that $(A-I)v_1 = 0$. Take $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The change of basis matrix is $P = (v_1 v_2) = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$, and $P^{-1}AP = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is upper triangular.

4. (17 points)

(a) Let V be a finite dimensional vector space, T, D, and N, three linear transformations in $L(V,V)$, such that $T = D + N$. State the three properties that D and N need to satisfy, in order for the above to be the Jordan decomposition of T.
Answer: i) D is diagonalizable, ii) N is nilpotent, iii) $DN = ND$.

(b) Let $A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, and $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note, that $P^{-1}AP = B$.

i. The Jordan decomposition of B is $B = D' + N'$, where $D' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $N' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

ii. The Jordan decomposition $A = D + N$ of A is:

$$A = PBP^{-1} = P(D' + N')P^{-1} = PD'P^{-1} + PN'P^{-1}.$$

We see that $D = PD'P^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $N = PN'P^{-1} = \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$.

iii. We verify that the matrices D and N in part 4(b)ii satisfy the properties in part 4a, by a direct calculation.

iv. Using Jordan decomposition of A we calculate:

$A^k = (D+N)^k = D^k + kD^{k-1}N + \ldots$, where the other terms involve powers N^i, $i \geq 2$, which vanish. Hence,

$$A^k = \begin{pmatrix} 2^k & 0 \\ 0 & 2^k \end{pmatrix} + k \begin{pmatrix} 2^{k-1} & 0 \\ 0 & 2^{k-1} \end{pmatrix} \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} (1-k)2^k & k2^{k+1} \\ -k2^{k-1} & (k+1)2^k \end{pmatrix}.$$

5. (17 points) Let V be an n-dimensional vector space over $\mathbb{R}$ with an inner product and u a unit vector in V. Recall, that the reflection R of V, with respect to the subspace $u^\perp$ orthogonal to u, is given by $R(v) = v - 2(v,u)u$.

   (a) $R^2(v) = R(R(v)) = R(v - 2(v,u)u) = [v - 2(v,u)u] - 2([v - 2(v,u)u], u)u = [v - 2(v,u)u] + 2(v,u)u = v$. Hence, $R^2 = 1$.

   (b) The minimal polynomial $m(x)$ of R divides $x^2 - 1 = (x-1)(x+1)$, since $R^2 - 1 = 0$. If $m(x) = x - 1$, then $R = 1$, which it is clearly not, by part 5d. If $m(x) = x + 1$, then $R = -1$, and the -1 eigenspace is the whole of V. This is the case, precisely if V is one dimensional, by part 5d. Hence, if $n = 1$, then $m(x) = x + 1$, and if $n \geq 2$, then $m(x) = x^2 - 1$.

   (c) R is diagonalizable, since its minimal polynomial is a product of linear terms with distinct roots.

   (d) We compute the -1 eigenspace of R by solving the equation $R(v) = -v$, which is equivalent to $v - 2(v,u)u = -v$. Solving for v in terms of u, we get that v is in the -1 eigenspace, if and only is $v = (v,u)u$, i.e., if and only if v is a scalar multiple of u.

   (e) The characteristic polynomial of R is $h(x) = (x-1)^{d_+}(x+1)^{d_-}$, where d_+ is the dimension of the $+1$ eigenspace $null(R-1)$ and d_- is the dimension of the -1 eigenspace $null(R+1)$, since R is diagonalizable. Now $d_- = 1$, by part 5d, and $d_+ + d_- = n$, since the characteristic polynomial has degree n. Hence, $d_+ = n - 1$ (We also saw in class several times, that the $+1$ eigenspace is $u^\perp$, which is $n-1$ dimensional). We conclude, that $h(x) = (x-1)^{n-1}(x+1)$.

   (f) The trace $tr(R)$ is the sum of the eigenvalues, repeated according to their multiplicity in the characteristic polynomial. Hence, $tr(R) = -1 + (n-1)1 = n - 2$.

6. (17 points) Let V be a finite dimensional vector space over a field F, and $T : V \to V$ a linear transformation.

   (a) Let $v \in V$ be an eigenvector of T with eigenvalue λ, and $g(x) = c_n x^n + \cdots + c_0$ a polynomial in $F[x]$. Show that v is an eigenvector of $g(T)$ and find its eigenvalue.

   Answer: $T^n(v) = \lambda^n v$. Hence,
   $g(T)v = c_n T^n(v) + \cdots + c_0 v = (c_n \lambda^n + \cdots + c_0)v = g(\lambda)v$. The eigenvalue is thus $g(\lambda)$.

   (b) Use part 6a to show, that every root of the characteristic polynomial $h(x)$ of T is also a root of the minimal polynomial $m(x)$ of T (without using the Cayley-Hamilton Theorem).

   Answer: Let λ be a root of the characteristic polynomial of T. Then there exists a non-zero eigenvector v with eigenvalue λ. Then $m(T)v = m(\lambda)v$, by the previous part. On the other hand, $m(T) = 0$, so $m(T)v = 0$. Hence, $m(\lambda)v = 0$, and thus $m(\lambda) = 0$.

7. (17 points) Let V be a four dimensional vector space over $\mathbb{C}$. Assume that the characteristic polynomial of T is $h(x) = (x - \lambda_1)^2(x - \lambda_2)^2$, and $\lambda_1 \neq \lambda_2$.

   (a) The four possible minimal polynomials $m(x)$ of T (with leading coefficient 1) are: $(x - \lambda_1)^{e_1}(x - \lambda_2)^{e_2}$, where $1 \leq e_1 \leq 2$, and $1 \leq e_2 \leq 2$, since $m(x)$ divides $h(x)$ and $m(x)$ and $h(x)$ have the same set of roots.

   (b) Assume that the minimal polynomial of T is $m(x) = (x - \lambda_1)^{e_1}(x - \lambda_2)^{e_2}$, set $V_i := null[(T - \lambda_i\mathbf{1})^{e_i}]$, where $\mathbf{1}$ is the identity transformation, and let $T_i \in L(V_i, V_i)$ be the restriction of T to V_i. Use the Primary Decomposition Theorem to show, that the minimal polynomial of T_i is $(x - \lambda_i)^{e_i}$. Hint: Show first that the minimal polynomial $m_i(x)$ of T_i divides $m(x)$ and the product $g(x) := m_1(x)m_2(x)$ satisfies $g(T) = 0$.

   Answer: Step 1 ($m_i(x)$ divides $m(x)$): Let v be a vector in V_i. Then $T_i(v) = T(v)$, by definition of T_i. So $m(T_i)v = m(T)v = 0$. Thus the minimal polynomial $m_i(x)$ of T_i divides $m(x)$.

   Step 2 ($g(T) = 0$): Let $\{v_1, \ldots, v_{n_1}\}$ be a basis of V_1, and $\{w_1, \ldots, w_{n_2}\}$ a basis of V_2. Their union is a basis of V, so it suffices to show, that $g(T)v_i = 0 = g(T)w_j$. Now $g(T)v_i = m_2(T)(m_1(T)v_i) = m_2(T)(0) = 0$. Similarly, $g(T)w_j = m_1(T)(m_2(T)w_j) = m_1(T)(0) = 0$.

   Step 3: $m(x)$ divides $m_1(x)m_2(x)$, since $m_1(T)m_2(T) = 0$, by Step 2. The product $m_1(x)m_2(x)$ divides $m(x)$, since each factor does, by Step 1, and the two factors are relatively prime (they do not have a common factor). Hence, $m(x) = m_1(x)m_2(x)$, since both sides have leading coeficient 1.

   (c) Assuming that the minimal polynomial of T is $(x - \lambda_1)^2(x - \lambda_2)$, the dimensions of the null spaces of $T - \lambda_1\mathbf{1}$, $(T - \lambda_1\mathbf{1})^2$, $T - \lambda_2\mathbf{1}$, and $(T - \lambda_2\mathbf{1})^2$ are: $\dim null(T - \lambda_1\mathbf{1}) = 1$, $\dim null[(T - \lambda_1\mathbf{1})^2] = 2$, $\dim null(T - \lambda_2\mathbf{1}) = 2$, and $\dim null[(T - \lambda_2\mathbf{1})^2] = 2$.

Reason: Let $V_1 := null[(T - \lambda_1\mathbf{1})^2]$ and $V_2 := null(T - \lambda_2\mathbf{1})$. Then $V = V_1 \oplus V_2$, by the Primary Decomposition Theorem, and $\dim(V_i) = 2$, for both $i = 1, 2$, by the Triangular Form Theorem, since the multiplicity of the corresponding eigenvalue, as a root of the characteristic polynomial, is 2. This explains two of the above four equalities.

The equality $\dim null(T - \lambda_1\mathbf{1}) = 1$: We know that $null(T - \lambda_1\mathbf{1})$ is non-zero, since λ_1 is an eigenvalue, and $null(T - \lambda_1\mathbf{1})$ is contained in $null[(T - \lambda_1\mathbf{1})^2]$, which is two dimensional. Hence, $1 \leq \dim null(T - \lambda_1\mathbf{1}) \leq 2$. If $\dim null(T - \lambda_1\mathbf{1}) = 2$, then we would have had a basis of V consisting of eigenvectors of T, T would have been diagonalizable, and the minimal polynomial would have been $(x - \lambda_1)(x - \lambda_2)$, which it is not.

The equality $\dim null[(T - \lambda_2\mathbf{1})^2] = 2$: This part was not needed for full credit. $\dim null[(T - \lambda_2\mathbf{1})^2] \geq \dim null(T - \lambda_2\mathbf{1}) = 2$. If strict inequality held, then there would have been a vector v in $null[(T - \lambda_2\mathbf{1})^2]$, which did not belong to $null(T - \lambda_2\mathbf{1})$. Then the order $m_v(x)$ of v would have been $(x - \lambda_2)^2$. This is impossible, since the order $m_v(x)$ divides the minimal polynomial.