Math 545 Solution of Midterm 1 Spring 2007

1. (10 points) Let V be an n-dimensional vector space over the field $\mathbb{R}$ and let $T : V \to \mathbb{R}^2$ be a linear transformation from V to $\mathbb{R}^2$. Prove that if T is not the zero transformation and T is not onto, then $\dim(null(T)) = n - 1$, where $null(T) := \{v \in V \ : \ T(v) = 0\}$.

   Answer: (Compare with problem 5 page 108 in the text). The subspace $T(V)$ of $\mathbb{R}^2$ has dimension $0 < \dim(T(V)) < 2$, since T is not the zero transformation and T is not onto. Hence $\dim(T(V)) = 1$. The Fundamental Theorem of Linear Algebra states, that

   $$\dim(null(T)) + \dim(T(V)) = \dim(V).$$

   Hence, $\dim(null(T)) = \dim(V) - \dim(T(V)) = n - 1$.

2. (10 points) Determine whether there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, satisfying $T(1, 1, 1) = (1, 2)$, $T(1, 2, 1) = (1, 1)$, and $T(2, 1, 2) = (2, 1)$. Justify your answer!

   Answer: (Compare with Problem 11 page 107 in the text). First check if there are any non-trivial linear relations among the three vectors in $\mathbb{R}^3$, by forming the 3×3 matrix A, with these vectors as columns, so that the coefficient vector of any linear relation among them is a solution of $Ax = 0$. Row reducing, we get:

   $$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   The general solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. We get the linear relation: $-3(1, 1, 1) + (1, 2, 1) + (2, 1, 2) = (0, 0, 0)$. If T exists and we apply T to both sides of the above relation, we get

   $$-3(1, 2) + (1, 1) + (2, 1) = (0, 0).$$

   The left hand side is $(0, -4)$, so such a T does not exist.

3. (20 points) Let V be the vector space of all polynomial functions

   $$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

   of degree ≤ 3 with real coefficients c_i, and $T : V \to V$ the linear transformation

   $$T(f) \quad = \quad (x + 1)\frac{\partial f}{\partial x} - f$$

   sending f to $(x + 1)$ times its derivative minus f itself.

   (a) (10 points) Find the matrix $[T]_\beta$ in the basis $\beta = \{1, x, x^2, x^3\}$ of V.

   Answer: (Compare with Problem 3 page 108 in the text).

   $$\begin{aligned} [T]_\beta &= \left([T(1)]_\beta [T(x)]_\beta [T(x^2)]_\beta [T(x^3)]_\beta\right) \\ &= \left([-1]_\beta \, [1]_\beta \, \left[2x + x^2\right]_\beta \, \left[3x^2 + 2x^3\right]_\beta\right) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

(b) (3 points) Find a basis for the null space $null(T) := \{f \ : \ T(f) = 0\}$. Justify your answer!

Answer: The row reduced echelon form of $[T]_\beta$ is $B := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The general solution of $Bx = 0$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, which is the subspace spanned by the coordinate vector $[1+x]_\beta$. Hence, $null(T)$ is spanned by $1+x$.

(c) (2 points) Determine the rank of T.

Answer: The rank of T is equal to the rank of its matrix $[T]_\beta$, which is equal to the rank of its reduced echelon form B, which is 3.

(d) (5 points) Find a basis for the image $T(V)$ of T (consisting of polynomials!!!).

Answer: The pivot columns of the reduced echelon form B are the first, third, and fourth. Hence, the first, third, and fourth columns of the matrix $[T]_\beta$ are a basis for the column space of $[T]_\beta$. Thus, $T(1)$, $T(x^2)$, $T(x^3)$ form a basis for $T(V)$. We calculated above that these three vectors are -1, $2x + x^2$, $3x^2 + 2x^3$. There are many other correct answers.

4. (20 points) Let V be a finite dimensional vector space over the real numbers, with an inner product. Recall that a linear transformation $T : V \to V$ is called an *orthogonal transformation*, if it preserves length, i.e., $\|T(v)\| = \|v\|$, for all $v \in V$.

(a) (10 points) Prove that the product TS, of two orthogonal transformations T and S, is an orthogonal transformation.

Answer: Compare with problem 7 page 129. $TS(v) = T(S(v))$, so $\|TS(v)\| = \|T(S(v))\| = \|S(v)\| = \|v\|$, where the second equality is due to the assumption, that T is orthogonal, and the last equality is due to S being orthogonal.

(b) (10 points) Let T be an orthogonal transformation of V. Show that $\det(T)$ is equal to 1 or -1.

Answer: Compare with problem 5 page 150. Let $\beta = \{u_1, \ldots, u_n\}$ be an orthonormal basis for V. The matrix $A := [T]_\beta$ satisfies $({}^tA)A = I$ (the transpose of A is equal to the inverse of A), by Theorem 15.11 page 127. Recall that $\det(A) = \det({}^tA)$. Hence,

$$1 = \det(I) = \det(({}^tA)A) = \det({}^tA)\det(A) = (\det(A))^2.$$

Thus, $\det(T) = \det(A) = \pm 1$.

5. (20 points) Let $v_1 = (1, 1, 0)$, $v_2 = (1, 0, 1)$, and $v_3 = (1, 1, 1)$.

(a) (8 points) Use the Gram-Schmidt process, and the above basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$, to find an orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$, such that $\text{span}\{v_1, \ldots, v_r\} = \text{span}\{u_1, \ldots, u_r\}$, for $1 \le r \le 3$.

Answer: (Compare with problem 1 page 129) $u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0)$.
$v_2 - (v_2, u_1)u_1 = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = \frac{1}{2}(1, -1, 2)$. Normalize to get

$u_2 = \frac{(1,-1,2)}{\|(1,-1,2)\|} = \frac{1}{\sqrt{6}}(1,-1,2)$.
$v_3 - (v_3,u_1)u_1 - (v_3,u_2)u_2 = (1,1,1) - (1,1,0) - \frac{1}{3}(1,-1,2) = \frac{1}{3}(-1,1,1)$.
$u_3 = \frac{(-1,1,1)}{\|(-1,1,1)\|} = \frac{1}{\sqrt{3}}(-1,1,1)$.

(b) (4 points) State the definition of an orthonormal basis, and check that the basis you found in part 5a is orthonormal.

Answer: An orthonormal basis for an n-dimensional vector space is a set of n vector $\{u_1,\dots,u_n\}$, satisfying $(u_i,u_i)=1$, for all i, and $(u_i,u_j)=0$, if $i \neq j$. Two points were given for the check of your answer in part 5a.

(c) (4 points) Find the distance from the vector v_3 to the plane spanned by $\{v_1,v_2\}$. (these vectors are given at the beginning of problem 5).

Answer: Compare with problem 13 page 131. The plane P spanned by $\{v_1,v_2\}$ is also spanned by $\{u_1,u_2\}$. *We need an orthonormal basis for P in order to compute the projection $\hat{v}_3 := (v_3,u_1)u_1 + (v_3,u_2)u_2$ of v_3 to P!!!* We get

$$\hat{v}_3 := (v_3,u_1)u_1 + (v_3,u_2)u_2 = (1,1,0) + \frac{1}{3}(1,-1,2) = \frac{2}{3}(2,1,1).$$

But, in fact, we need the difference $v_3 - \hat{v}_3$, which was already calculated in part 5a. The distance is $\|v_3 - \hat{v}_3\| = \|\frac{1}{3}(-1,1,1)\| = \frac{1}{3}\|(-1,1,1)\| = \frac{1}{\sqrt{3}}$.

(d) (4 points) Explain how to read, from the orthonormal basis you found in part 5a, without any further computations, the equation of the plane spanned by $\{v_1,v_2\}$.

Answer: Compare with part (a) of problem 11 page 130. The vector u_3 is orthogonal to the plane P spanned by $\{u_1,u_2\}$, which is equal to the plane spanned by $\{v_1,v_2\}$. Hence the plane P is equal to $\{v : (v,u_3)=0\}$. Using dot product, it becomes

$$P = \{(x_1,x_2,x_3) \; : \; \frac{1}{\sqrt{3}}(-x_1+x_2+x_3)=0\},$$

or simply $-x_1+x_2+x_3=0$.

6. (20 points) Let V be an n-dimensional vector space with an inner product and u a unit vector in V (so that $(u,u)=1$). Let $u^\perp$ be the subspace $\{v \in V \; : \; (v,u)=0\}$, orthogonal to u. Recall that the reflection $R_u : V \to V$, of V with respect to $u^\perp$, is given by

$$R_u(v) \;=\; v - 2(v,u)u.$$

(a) (8 points) Prove that R_u is a *linear* transformation (it is also easy to show that R_u is an orthogonal transformation, but you are not asked to show it).

Answer: Check the two properties in the definition of a linear transformation: 1) For every two vectors $v_1,v_2 \in V$ we have

$$\begin{aligned} R_u(v_1+v_2) &= v_1+v_2-2(v_1+v_2,u)u = v_1+v_2-2(v_1,u)u-2(v_2,u)u \\ &= [v_1-2(v_1,u)u]+[v_2-2(v_2,u)u] = R_u(v_1)+R_u(v_2). \end{aligned}$$

2) For every $\lambda \in \mathbb{R}$ and every $v \in V$, we have

$$R_u(\lambda v) = \lambda v - 2(\lambda v,u)u = \lambda v - 2\lambda(v,u)u = \lambda[v-2(v,u)u] = \lambda R_u(v).$$

(b) (4 points) Let u_1 and u_2 be two unit vectors in V. Show that if $(u_1, u_2) = 0$, then $R_{u_1}R_{u_2} = R_{u_2}R_{u_1}$. In other words, the two reflections commute, if the two unit vectors are orthogonal.

Answer: We need to prove the equality $R_{u_1}R_{u_2}(v) = R_{u_2}R_{u_1}(v)$ for every vector v in V.

$$\begin{aligned} R_{u_1}R_{u_2}(v) &= R_{u_1}(R_{u_2}(v)) = R_{u_1}\left(v - 2(v,u_2)u_2\right) = \\ &= [v - 2(v,u_2)u_2] - 2\left([v - 2(v,u_2)u_2], u_1\right) u_1 \\ &= v - 2(v,u_2)u_2 - 2(v,u_1)u_1 + 4(v,u_2)(u_2,u_1)u_1 \\ &= v - 2(v,u_2)u_2 - 2(v,u_1)u_1. \end{aligned}$$

The last equality uses the vanishing $(u_2, u_1) = 0$. Now the last term we got is symmetric in u_1 and u_2 and so is equal also to $R_{u_2}R_{u_1}(v)$.

(c) (8 points) Let $V = \mathbb{R}^2$, with the standard inner product (the dot product), and set $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Find the matrix $[R_u]_\beta$, of the reflection R_u, with respect to the basis $\beta = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\ (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$. Justify your answer!

Answer: Compare with part b.iii of the additional problem to section 18. (Parts b.i and b.ii of that problem were added after the exam for future semesters). Set $u_1 := (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $u_2 := (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Then
$R_u(u_1) = u_1 - 2(u_1, u)u = u_1 - 2(u_1, u_1)u_1 = -u_1$ and
$R_u(u_2) = u_2 - 2(u_2, u)u = u_2 - 0 = u_2$. We get

$$[R_u]_\beta = ([R_u(u_1)]_\beta [R_u(u_2)]_\beta) = ([-u_1]_\beta [u_2]_\beta) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$