Math 545 Midterm 2 Spring 2007

Solve 6 out of the following 7 problems.

Show all your work and justify all your answers!!!

1. (17 points) Let θ be an angle, such that $\sin(\theta) \neq 0$, and let $A := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ be the matrix of the rotation of $\mathbb{R}^2$ about the origin by angle θ counterclockwise.

   (a) Find the characteristic polynomial of A.

   (b) Find the minimal polynomial of A.

   (c) Show that A is similar to a diagonal matrix in $M_2(\mathbb{C})$.

   (d) Show that A is not similar to a diagonal matrix in $M_2(\mathbb{R})$.

2. (17 points) Set $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.

   (b) Find a basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

   (c) Find an invertible matrix P and a diagonal matrix D, both in $M_2(\mathbb{C})$, such that $P^{-1}AP = D$.

3. (17 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by multiplication by $A = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$.

   (a) Find the characteristic polynomial of T.

   (b) Find the minimal polynomial of T.

   (c) Determine if T is diagonalizable.

   (d) Find the eigenvalues of T.

   (e) Find a basis for each eigenspace of T.

   (f) Find an upper triangular matrix B and an invertible matrix P, such that $B = P^{-1}AP$. Carefully explain, in complete sentences, your method for finding P. Credit will not be given for an answer obtained by trial and error.

4. (17 points)

   (a) Let V be a finite dimensional vector space, T, D, and N, three linear transformations in $L(V, V)$, such that $T = D + N$. State the three properties that D and N need to satisfy, in order for the above to be the Jordan decomposition of T.

   (b) Let $A = \begin{pmatrix} 0 & 4 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, and $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note, that $P^{-1}AP = B$.

      i. Find the Jordan decomposition of B.

      ii. Find the Jordan decomposition $A = D + N$ of A.

      iii. Check directly that the matrices D and N you found in part 4(b)ii satisfy the properties in part 4a.

iv. Use the Jordan decomposition of A to calculate the entries of A^k, as functions of k, for all positive integers k.

5. (17 points) Let V be an n-dimensional vector space over $\mathbb{R}$ with an inner product and u a unit vector in V. Recall, that the reflection R of V, with respect to the subspace $u^\perp$ orthogonal to u, is given by

$$R(v) \quad = \quad v - 2(v, u)u.$$

   (a) Show that $R^2 = 1$.
   (b) Find the minimal polynomial of R. Justify your answer.
   (c) Show that R is diagonalizable.
   (d) Show that the -1 eigenspace of R is spanned by u.
   (e) Find the characteristic polynomial of R. Justify all your answers!
   (f) Calculate the trace $tr(R)$.

6. (17 points) Let V be a finite dimensional vector space over a field F, and $T : V \to V$ a linear transformation.

   (a) Let $v \in V$ be an eigenvector of T with eigenvalue λ, and $g(x) = c_n x^n + \cdots + c_0$ a polynomial in $F[x]$. Show that v is an eigenvector of $g(T)$ and find its eigenvalue.
   (b) Use part 6a to show, that every root of the characteristic polynomial $h(x)$ of T is also a root of the minimal polynomial $m(x)$ of T (without using the Cayley-Hamilton Theorem).

7. (17 points) Let V be a four dimensional vector space over $\mathbb{C}$. Assume that the characteristic polynomial of T is $(x - \lambda_1)^2(x - \lambda_2)^2$, and $\lambda_1 \neq \lambda_2$.

   (a) What are all the possible minimal polynomials $m(x)$ of T (with leading coefficient 1)? Justify your answer!
   (b) Assume that the minimal polynomial of T is $m(x) = (x - \lambda_1)^{e_1}(x - \lambda_2)^{e_2}$, set $V_i := null[(T - \lambda_i \mathbf{1})^{e_i}]$, where $\mathbf{1}$ is the identity transformation, and let $T_i \in L(V_i, V_i)$ be the restriction of T to V_i. Use the Primary Decomposition Theorem to show, that the minimal polynomial of T_i is $(x - \lambda_i)^{e_i}$. Hint: Show first that the minimal polynomial $m_i(x)$ of T_i divides $m(x)$ and the product $g(x) := m_1(x)m_2(x)$ satisfies $g(T) = 0$.
   (c) Assume that the minimal polynomial of T is $(x - \lambda_1)^2(x - \lambda_2)$. Calculate the dimensions of the null spaces of $T - \lambda_1 \mathbf{1}$, $(T - \lambda_1 \mathbf{1})^2$, $T - \lambda_2 \mathbf{1}$, and $(T - \lambda_2 \mathbf{1})^2$. Carefully explain how your answer follows from the Primary Decomposition Theorem and the Triangular Form Theorem.