Math 421 Final Exam Spring 2005

Solve problem 1 and only 7 out of problems 2 to 9. If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number _____. Show **all** your work. Credit will **not** be given for an answer without a justification. Calculators may **not** be used in this exam.

1. (16 points) Given that the Taylor series of $\tan(z)$, centered at 0, has the form

$$\tan(z) \;=\; z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots \text{terms of order at least seven.} \qquad (1)$$

   a) Evaluate the fifth derivative $\tan^{(5)}(0)$ with as little calculations as possible.

   b) Find the principal part at $z = 0$ of the function $f(z) \;=\; \dfrac{(1+z)\tan(z)}{z^5}$

   c) Find all the singularities of $f(z)$ (given in part b) in the disk $D = \{|z| < 4\}$ and determine their type (isolated, removable, pole of what order, essential).

   d) Find the residue at each isolated singularity in D.

2. (12 points) a) Compute $\sin(\pi + i\ln(3))$. Simplify your answer as much as possible.

   b) Prove that all solutions of the equation $\cos(z) = 0$ are real and find all the solutions.

3. (12 points) Compute the integral $\displaystyle\int_C \frac{z^5}{1-z^3}dz$, where C is the circle of radius 2, centered at 0, and traversed counterclockwise.

4. (12 points) a) Find the Taylor series of the function $f(z) = \dfrac{2z+1}{z^2+z-2} = \dfrac{1}{z-1} + \dfrac{1}{z+2}$ centered at 0 and determine its radius of convergence. Justify your answer.

   b) Find the Laurent series of the function $f(z)$, given in part a), valid in the annulus $1 < |z| < 2$.

5. (12 points) a) Use the definition of contour integrals to prove the equality

$$\int_C \sin(\bar{z})dz \;=\; \int_C \sin(1/z)dz, \qquad (2)$$

   where C is the circle $\{z \;:\; |z| = 1\}$, traversed counterclockwise. *Caution: The argument of the integrand, on the left hand side, is the complex conjugate $\bar{z}$ of z.*

   b) Find the Laurent series of $\sin(1/z)$ centered at zero and classify the type of singularity at $z = 0$.

   c) Use the equality (2) in order to evaluate the integral $\displaystyle\int_C \sin(\bar{z})dz$.

6. (12 points) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)}.$$

   Show all your work!

7. (12 points) Let S_R be the upper-semi-circle of radius $R > 1$, given by the parametrization $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Prove the equality

$$\lim_{R\to\infty} \int_{S_R} \frac{z^2 dz}{1+z^4} \;=\; 0.$$

Hint: Find first an upper bound for the integral.

8. (12 points) Determine whether the following statements are true or false. Justify your answers!

   a) Let C be the circle $\{z \;:\; |z| = 1\}$, oriented counterclockwise. Assume that $f(z)$ is analytic in the punctured disk $0 < |z| < 2$, and the integrals $\int_C z^n f(z) dz$ vanish, for all integers $n \geq 0$. Then 0 is a removable singularity of f.

   b) There exists a function $F(z)$, analytic in the punctured unit disk $\{z \;:\; 0 < |z| < 1\}$, whose derivative $f(z) := F'(z)$ satisfies $Res_{z=0}(f(z)) = 1$.

   c) If f is a non-constant entire function and $|f(z)| \leq 2$, for every z on the unit circle $\{z \;:\; |z| = 1\}$, then f must map the unit disk $\{z \;:\; |z| < 1\}$ into the disk $\{z \;:\; |z| < 2\}$.

   d) There exists an entire function, whose real part is e^{x+y}.

9. (12 points) Evaluate the improper integral

$$\int_0^\infty \frac{x^2}{x^4+1} dx.$$