Math 421 Midterm 1 Solution Fall 2002

1. (36 points) Let $z = \dfrac{10}{\sqrt{3}-i}$. Compute the following (in cartesian or polar form):

   a) The polar form of z is

$$z = 5\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \;=\; 5e^{i\frac{\pi}{6}}.$$

   b) $|z^3| = |z|^3 = 5^3 = 125.$

   c) $\mathrm{Log}(z^9) \;=\; \ln\left|z^9\right| + i\mathrm{Arg}(z^9) \;=\; 9\ln(5) + i\mathrm{Arg}(e^{\frac{9\pi\cdot i}{6}}) \;=\; 9\ln(5) - \frac{\pi}{2}i$

   d) The three values of $z^{\frac{1}{3}}$ are:

$$z^{\frac{1}{3}} \;=\; 5^{1/3}e^{\frac{\pi i}{18}} \cdot e^{\frac{2n\pi i}{3}},$$

   where $n = 0, 1, 2$. More explicitly, they are

$$5^{1/3}e^{\frac{\pi i}{18}}, \quad 5^{1/3}e^{\frac{13\pi i}{18}}, \quad \text{and} \quad 5^{1/3}e^{\frac{25\pi i}{18}}.$$

   e) All values of z^{2i} are:

$$z^{2i} \;=\; e^{2i\log(z)} \;=\; e^{2i\log(5e^{i\frac{\pi}{6}})} \;=\; e^{2i[\ln(5)+i(\frac{\pi}{6}+2n\pi)]} \;=\; e^{-[\frac{\pi}{3}+4n\pi]}\cdot e^{2\ln(5)i},$$

   where n is an integer.

2. (18 points) a) Find the image, under the principal branch of $\mathrm{Log}(z)$, of the set

$$\{z \text{ such that } |z| = 2 \text{ and } z \neq -2\}$$

   (circle of radius 2, with the point -2 removed).

   Answer: $\mathrm{Log}(2e^{i\theta}) = \ln(2) + i\theta$, if $-\pi < \theta < \pi$. Thus, the image set is:

$$\{(\ln(2) + i\theta), \text{ where } -\pi < \theta < \pi\}$$

   This is an interval, on the vertical line $x = \ln(2)$, with y-coordinate between $-\pi$ and π.

   b) Find the image of the vertical line $x = 2$ under the function $f(z) = e^{iz}$.

   Answer: $e^{i(2+iy)} = e^{-y}\cdot e^{2i}$. Thus, the image set is:

$$\{e^{-y}\cdot e^{2i}, \text{ where } y \text{ is real}\}.$$

   This is the open ray (since e^{-y} is any positive number) with angle 2 radians, on the line passing through the origin.

3. (18 points) a) $\cos(i) \;=\; \dfrac{e^{i^2}+e^{-(i^2)}}{2} = \dfrac{e^{-1}+e}{2}.$

b) Find all solutions of the equation $\cos(z) = 10$.

Answer: We need to solve the equation

$$\frac{e^{iz}+e^{-(iz)}}{2} \;=\; 10.$$

Substitute $w = e^{iz}$ and multiply both sides by 2 to get

$$w + \frac{1}{w} \;=\; 20.$$

The latter is equivalent to the quadratic equation

$$w^2 - 20w + 1 \;=\; 0.$$

Its two solutions are

$$w_{1,2} \;=\; \frac{20 \pm \sqrt{400-4}}{2} \;=\; 10 \pm \sqrt{99}.$$

Expressing the solution in terms of z requires solving the two equations

$$e^{iz} \;=\; 10 \pm \sqrt{99}.$$

We get

$$z \;=\; -i\log(10+\sqrt{99}) \;=\; -i\ln(10+\sqrt{99}) + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

and

$$z \;=\; -i\log(10-\sqrt{99}) \;=\; -i\ln(10-\sqrt{99}) + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

4. (18 points) a) Prove that the function

$$u(x,y) \;=\; x^3 - 3xy^2 - 2x + e^{-y}\cos(x)$$

is harmonic on the whole of $\mathbb{R}^2$.

Answer: Method 1: A short answer to all three parts can be obtained by solving part (c) first. One then answer part (a) using the Theorem, that the real part of an analytic function is a harmonic function. Part (b) can be solved by calculating the imaginary part of the entire function in part (c).

Method 2: We need to prove that u satisfies the Laplace equation $u_{xx} + u_{yy} = 0$. We calculate

$$\begin{aligned} u_x &= 3x^2 - 3y^2 - 2 - e^{-y}\sin(x) && (1)\\ u_{xx} &= 6x - e^{-y}\cos(x), \\ u_y &= -6xy - e^{-y}\cos(x), && (2)\\ u_{yy} &= -6x + e^{-y}\cos(x). \end{aligned}$$

The Laplace equation is obviously satisfied.

b) A harmonic conjugate v of the function u can be calculated as follows:

$$\begin{aligned} v(x,y) &= \int v_x dx + h(y) = \int -u_y dx + h(y) \overset{(2)}{=} \int 6xy + e^{-y}\cos(x) dx + h(y) = \\ &= 3x^2 y + e^{-y}\sin(x) + h(y), \end{aligned}$$

where $h(y)$ is a function of y (independent of x). We determine $h(y)$ using the equation $v_y = u_x$. Using (1) we get

$$h'(y) = -3y^2 - 2$$

and $h(y) = -y^3 - 2y + C$. Summarizing,

$$v(x,y) = 3x^2 y + e^{-y}\sin(x) - y^3 - 2y + C.$$

c) An entire function $f(z)$, such that $Re(f) = u$, is

$$f(z) = z^3 - 2z + e^{iz}.$$

5. (10 points) Let $f(z)$ be an entire function, whose real and imaginary parts satisfy the following relation

$$\mathrm{Re}(f) = 2\mathrm{Im}(f). \tag{3}$$

Prove that f must be a constant function. *Hint: Use the Cauchy-Riemann equations to prove that $f'(z) = 0$.*

Answer: Write $f = u + iv$. The Cauchy-Riemann equations are

$$u_x = v_y, \tag{4}$$
$$u_y = -v_x. \tag{5}$$

Equation (3) yields

$$u_x = 2v_x, \tag{6}$$
$$u_y = 2v_y. \tag{7}$$

The four equations above is a system of four linear homogeneous equations, in the four "unknowns" u_x, u_y, v_x, v_y. It has only the trivial solution, namely

$$u_x = u_y = v_x = v_y = 0.$$

You can see that by raw reduction, or as follows:

$$u_x \overset{(6)}{=} 2v_x \overset{(5)}{=} -2u_y \overset{(7)}{=} -4v_y \overset{(4)}{=} -4u_x.$$

Thus, $5u_x = 0$ and $u_x = 0$. By (6), $v_x = 0$. By (4), $v_y = 0$. By (5), $u_y = 0$.

We conclude, that the partials of u and v vanish. By a theorem from third semester calculus, u and v must be constant functions on the whole plane.