Math 421 Midterm 1 Spring 2005

Name:__________________________

1. (36 points) Let $z = \dfrac{6}{\sqrt{2} - \sqrt{2}i}$. Compute the following (in cartesian or polar form):

   a) The polar form of z.

   b) $|z^3|$

   c) $\mathrm{Log}(z^6)$

   d) All values of $z^{\frac{1}{5}}$. How many different values are there?

   e) All values of z^i. How many different values are there?

2. (10 points) Let $f(z)$ be an entire function satisfying $|f(z)|^2 = 2$ for all z. Prove that f must be a constant function. *Hint: Show that the conjugate function $\overline{f(z)}$ must be entire. Then use the Cauchy-Riemann equations to prove that $f'(z) = 0$.*

3. (18 points) a) Compute the Cartesian coordinates of $\sin(2i)$.

   b) Find the set of points in the plane, where the function $\dfrac{z}{\sin(z) - 2i\cos(z)}$ is differentiable. Justify your answer!

4. (18 points) a) Prove that the function

   $$u(x, y) \;=\; e^x \sin(y) + e^y \cos(x) + 2xy$$

   is harmonic on the whole of $\mathbb{R}^2$.

   b) Find a harmonic conjugate v of the function u.

   c) Find an entire function $f(z)$ such that $Re(f) = u$. Your answer must be expressed as a function of $z = x + iy$, not x and y.

5. (18 points) a) Find the image of the horizontal line $y = 1/4$ under the function $f(z) = e^{\pi z}$.

   b) Find the image, under the principal branch of $\mathrm{Log}(z)$, of the set

   $$\{z \text{ such that } |z| < 1 \text{ and } \mathrm{Re}(z) > 0\}$$

   (the right half of the unit disk).