Math 421 Midterm 1 Solution Spring 2005

1. (36 points) Let $z = \dfrac{6}{\sqrt{2}-\sqrt{2}i}$. Compute the following (in cartesian or polar form):

   a) (8 points) The polar form of z is $z = \dfrac{6(\sqrt{2}+\sqrt{2}i)}{(\sqrt{2})^2+(\sqrt{2})^2} = 3e^{\pi i/4}$.

   b) (7 points) $|z^3| = |z|^3 = 3^3 = 27$.

   c) (7 points) $\mathrm{Log}(z^6) = \mathrm{Log}(3^6 e^{3\pi i/2}) = 6\ln(3) - i\pi/2$

   d) (7 points) The five values of $z^{\frac{1}{5}}$ are: $3^{1/5} \cdot e^{i[\pi/20+2n\pi/5]}$, where $n = 0, 1, 2, 3, 4$.

   e) (7 points) The values of z^i are:
   $(3e^{i\pi/4})^i = e^{i[\log(3e^{i\pi/4})]} = e^{i[\ln(3)+i(\pi/4+2n\pi i)]} = e^{-\pi/4-2n\pi+i\ln(3)}$, where n is an integer. There are infinitely many such values.

2. (10 points) Let $f(z)$ be an entire function satisfying $|f(z)|^2 = 2$ for all z. Prove that f must be a constant function. *Hint: Show that the conjugate function $\overline{f(z)}$ must be entire. Then use the Cauchy-Riemann equations to prove that $f'(z) = 0$.*

   Method 1: We first show that $\overline{f(z)}$ is entire (analytic at every point of the plane): The equality $2 = |f(z)|^2 = f(z) \cdot \overline{f(z)}$ implies, that $\overline{f(z)} = 2/f(z)$ and $f(z)$ does not vanish. By the quotient rule of differentiation, $2/f(z)$ is entire.

   Write $f(z) = u(x,y) + iv(x,y)$. Then $\overline{f(z)} = u(x,y) - iv(x,y)$. The Cauchy-Riemann equations (1), (2) for $f(z)$ and (3), (4) for $\overline{f(z)}$ are:

$$u_x = v_y \quad \text{and} \tag{1}$$
$$u_y = -v_x, \tag{2}$$
$$u_x = (-v_y) \quad \text{and} \tag{3}$$
$$u_y = -(-v_x). \tag{4}$$

   Equations (1) and (3) imply that $u_x = u_y = 0$. Equations (2) and (4) imply that $u_y = v_x = 0$. We conclude that u and v, and hence also f, are constant functions.

   Method 2: (without showing that $\overline{f(z)}$ is entire). Write $f(z) = u(x,y) + iv(x,y)$. Diffrerentiate both sides of the given equality $u^2 + v^2 = 2$ to get: $2uu_x + 2vv_x = 0$ and $2uu_y + 2vv_y = 0$. Use the Cauchy-Riemann equations to replace the second equality by an equality involving the partials u_x and v_x. We get the system of two linear equations in u_x and v_x:

$$\begin{aligned} 2uu_x + 2vv_x &= 0, \\ 2vu_x - 2uv_x &= 0, \end{aligned}$$

   whose matrix $\begin{bmatrix} 2u & 2v \\ 2v & -2u \end{bmatrix}$ has determinant $-4(u^2+v^2) = -8 \neq 0$. Hence, the system has only the trivial solution $u_x = v_x = 0$. The Cauchy-Riemann equations imply also that $u_y = v_y = 0$ and hence f is a constant function.

3. a) (6 points) $\sin(2i) = \dfrac{e^{i(2i)} - e^{-i(2i)}}{2i} = \dfrac{e^{-2} - e^{2}}{2i} = \left[\dfrac{-e^{-2}+e^{2}}{2}\right] i.$

b) (12 points) Find the set of points in the plane, where the function $f(z) := \dfrac{z}{\sin(z) - 2i\cos(z)}$ is differentiable. Justify your answer!

Answer: The functions z, $\sin(z)$, and $\cos(z)$ are entire. Hence, $\sin(z) - 2i\cos(z)$ is entire, and the quotient $f(z)$ is analytic at every point, where the denominator does not vanish (the quotient differentiation rule). The points where the denominator vanishes are the solution of $\sin(z) = 2i\cos(z)$, which, by definition, is

$$\frac{e^{iz} - e^{-iz}}{2i} = 2i\frac{e^{iz} + e^{-iz}}{2}.$$

Multimply both sides by $2i$ and collect the terms to get $3e^{iz} = -e^{-iz}$.
Multiply both sides by e^{iz} to get $e^{2iz} = -1/3 = 1/3e^{\pi i}$. Set $z := x + iy$. We get

$$\begin{aligned} e^{-2y+2ix} &= 1/3e^{\pi i}, \quad \text{or} \\ -2y = \ln(1/3) \quad &\text{and} \quad 2ix = i\pi + 2n\pi i. \end{aligned}$$

The general solution is $y = \ln(3)/2$ and $x = \frac{\pi}{2} + n\pi$, where n in an integer.

4. a) (6 points) The function $u(x, y) = e^x \sin(y) + e^y \cos(x) + 2xy$ is harmonic on the whole of $\mathbb{R}^2$, because it satisfies the Laplace equation $u_{xx} + u_{yy} = 0$. We verify this by plugging in (or by answering part (c) first).

$$\begin{aligned} u_x &= e^x \sin(y) - e^y \sin(x) + 2y, \\ u_{xx} &= e^x \sin(y) - e^y \cos(x), \\ u_y &= e^x \cos(y) + e^y \cos(x) + 2x, \\ u_{yy} &= -e^x \sin(y) + e^y \cos(x). \end{aligned}$$

b) (8 points) The harmonic conjugate v of the function u satisfies the two Cauchy-Rieman equations $v_x = -u_y$ and $v_y = u_x$. Integrating the second, we get:
$v(x, y) = \int u_x dy = \int [e^x \sin(y) - e^y \sin(x) + 2y]dy = -e^x \cos(y) - e^y \sin(x) + y^2 + h(x)$.
We find $h'(x)$ using the first equation $v_x = -u_y$:
$-e^x \cos(y) - e^y \cos(x) + h'(x) = -[e^x \cos(y) + e^y \cos(x) + 2x]$.
Hence, $h'(x) = -2x$, $h(x) = -x^2 + C$, and the harmonic conjugate is:
$v(x, y) = -e^x \cos(y) - e^y \sin(x) + y^2 - x^2 + C$.

c) (4 points) Find an entire function $f(z)$ such that $Re(f) = u$.

Answer: $f(z) = -ie^z + e^{-iz} - iz^2$.

5. a) (9 points) The general point on the horizontal line $y = 1/4$ has the form $z = x + (1/4)i$. The function $f(z) = e^{\pi z}$ takes this point to $e^{\pi x + \pi i/4} = e^{\pi x} \cdot \dfrac{1+i}{\sqrt{2}}$. The image of the line $y = 1/4$ under the function $f(z) = e^{\pi z}$ is the half-line obtained by multiplying $1 + i$ by an arbitrary positive real number, namely the half-line with angle $\pi/4$.

b) (9 points) Find the image, under the principal branch of $\mathrm{Log}(z)$, of the set $\{z \text{ such that } |z| < 1 \text{ and } \mathrm{Re}(z) > 0\}$ (the right half of the unit disk).

Answer: A general point in this half-disk has the form $z = re^{i\theta}$, where $0 < r < 1$, and $-\pi/2 < \theta < \pi/2$. Hence $Log(z) = \ln(r) + i\theta$, where $\ln(r) < 0$. The image is the left-half strip: $\left\{x + iy \quad \text{such that} \quad x < 0 \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$.