Math 421 Midterm 1 solution Fall 2009

1. (36 points) Compute the following (in cartesian or polar form):

   a) Compute the polar form of $z = \dfrac{8}{-\sqrt{2}+\sqrt{2}i}$.

   Answer: $4e^{-i(3\pi/4)}$.

   b) $|z^{-2}|$, where z is given in part a.

   Answer: $\frac{1}{16}$.

   c) $\text{Log}(a^3)$, where $a = 2e^{i[4\pi/5]}$.

   Answer: $a^3 = 8e^{i(12\pi/5)} = 8e^{i(2\pi/5)}$. Hence, $Log(a^3) = 3\ln(2) + i\frac{2\pi}{5}$.

   d) Find all values of $(1-i)^{\frac{1}{4}}$. How many different values are there?

   Answer: $(1-i) = \sqrt{2}e^{-i(\pi/4)}$. Any non-zero complex number $re^{i\theta}$ has 4 distinct fourth-roots. One fourth root is $w_0 = r^{1/4}e^{i\theta/4}$, and all four roots are $w_k = w_0 e^{(i(k\pi/2)}$, $k = 0,1,2,3$. Hence, one fourth root is $w_0 = 2^{1/8}e^{-i(\pi/16)}$. The other three are: $w_1 = 2^{1/8}e^{i(7\pi/16)}$, $w_2 = 2^{1/8}e^{i(15\pi/16)}$, $w_1 = 2^{1/8}e^{i(23\pi/16)}$.

   e) Find all values of $i^{[(1-i)/2]}$. How many different values are there?

   Answer: $i^{[(1-i)/2]} = e^{\log(i)[(1-i)/2]} = e^{\left[(\frac{\pi}{4}+k\pi)+i(\frac{\pi}{4}+k\pi)\right]} = (-1)^k e^{\left[(\frac{\pi}{4}+k\pi)+i(\frac{\pi}{4})\right]}$. There are infinitely many different values, as the absolute-values (modulii) $e^{\left(\frac{\pi}{4}+k\pi\right)}$ are distinct, for distinct integral values of k.

2. (18 points) Determine which of the following functions is entire (analytic on the whole complex plane). Prove your answer. Carefully state each theorem you are using.
   a) $f(z) = x^2 + y^2 + i(2xy)$.

   Answer: The function is not analytic, hence not entire. We will use the following theorem with $U = \mathbb{C}$, $u(x,y) = x^2 + y^2$, and $v(x,y) = 2xy$.
   Theorem: *Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function on an open set U of the complex plane. Then the partials of u and v exist in U and satify the Cauchy-Riemann equations $u_x = v_y$, and $u_y = -v_x$, at every point of U.*

   Now $u_y = 2y$, $v_x = 2y$, and so $u_y \neq -v_x$, if $y \neq 0$. Hence, the second Cauchy-Riemann equation is not satisfied throught $\mathbb{C}$.

b) $f(z) = e^{(x-y)}\sin(x+y) - ie^{(x-y)}\cos(x+y)$.

Answer: The function is entire. We will use the following theorem with $U = \mathbb{C}$, $u(x,y) = e^{(x-y)}\sin(x+y)$, and $v(x,y) = -e^{(x-y)}\cos(x+y)$.

Theorem: *Let u and v be real valued functions defined on an open set U of the complex plane. Assume that the partials u_x, u_y, v_x and v_y, all exist and are continuous in U. Assume further that the partials satisfy the Cauchy-Riemann equations $u_x = v_y$, and $u_y = -v_x$, at every point of U. Then the function $f(z) = u(x,y) + iv(x,y)$, with $z = x + iy$, is analytic in U.*

The partials

$$\begin{aligned} u_x(x,y) &= e^{(x-y)}[\sin(x+y) + \cos(x+y)], \\ u_y(x,y) &= e^{(x-y)}[-\sin(x+y) + \cos(x+y)], \\ v_x(x,y) &= e^{(x-y)}[\sin(x+y) - \cos(x+y)], \\ v_y(x,y) &= e^{(x-y)}[\sin(x+y) + \cos(x+y)]. \end{aligned}$$

indeed exist and are continuous in $\mathbb{C}$, and they satisfy the Cauchy-Riemann equations.

A short-cut: Once one recognizes that $f(z) = -ie^{z+iz}$, one can use the fact that e^z is analytic, plus the lemma, which states that the composition of analytic functions is analytic. This method was attempted by a few students with partial success.

3. (10 points) Compute the Cartesian coordinates of $\cos\left(\frac{\pi}{4} - \frac{i}{2}\ln(2)\right)$. Show all your work and simplify your answer as much as possible.

   Answer: Recall that $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$. Using the identity $e^{a\ln(b)} = (e^{\ln(b)})^a = b^a$, for all real numbers a and all positive real numbers b, we get $e^{\ln(2)/2} = \sqrt{2}$.

   $$\cos\left(\frac{\pi}{4} - \frac{i}{2}\ln(2)\right) = \frac{1}{2}\left[e^{\ln(2)/2+\frac{i\pi}{4}} + e^{-\ln(2)/2-\frac{i\pi}{4}}\right] =$$
   $$\frac{1}{2}\left[\sqrt{2}e^{i\pi/4} + \frac{1}{\sqrt{2}}e^{-i\pi/4}\right] = \frac{1}{2}\left[\sqrt{2}\frac{1+i}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1-i}{\sqrt{2}}\right] = \frac{3}{4} + \frac{i}{4}.$$

4. (18 points) a) Prove that the function

   $$u(x,y) = y^3 - 3x^2y + 2x^2 - 2y^2 + e^x\sin(y)$$

   is harmonic on the whole of $\mathbb{R}^2$.

   Answer: A function $u(x,y)$ is Harmonic, if its second partials u_{xx}, u_{xy}, u_{yx}, u_{yy}, exist and are continuous, and if it satisfies the Laplace equation $u_{xx} + u_{yy} = 0$. In our case we have

   $$\begin{aligned} u_x &= -6xy + 4x + e^x\sin(y), & (1) \\ u_{xx} &= -6y + 4 + e^x\sin(y), \\ u_y &= 3y^2 - 3x^2 - 4y + e^x\cos(y), & (2) \\ u_{yy} &= 6y - 4 - e^x\sin(y). \end{aligned}$$

   All partials are continuous, being sums of polynomials in x and y and of constant multiples of $e^x\cos(y)$ and $e^x\sin(y)$. The Laplace equation is satisfied. Hence u is Harmonic.

b) Find a harmonic conjugate v of the function u.

Answer: By definition, v is the harmonic function, such that $f(z) = u(x, y) + iv(x, y)$ is analytic. Hence, the partials of u and v should satisfy the Cauchy-Riemann equations. Use the Cauchy-Riemann equation $v_x = -u_y$ to find v, up to a summand involving a function of y, by integration:
$v(x, y) = \int v_x dx = \int [-u_y] dx \overset{(2)}{=} -\int [3y^2 - 3x^2 - 4y + e^x \cos(y)] dx =$
$-3xy^2 + x^3 + 4xy - e^x \cos(y) + h(y)$.
We find $h'(y)$ by comparing the derivative of the above result to v_y. The partial v_y is known, by the Caushy-Riemann equations, to be equal to u_x given in equation (1). We get

$$-6xy + 4x + e^x \sin(y) + h'(y) = v_y \overset{C.R.}{=} u_x \overset{(1)}{=} -6xy + 4x + e^x \sin(y).$$

Hence, $h(y)$ is constant, and we may choose $v(x, y) = -3xy^2 + x^3 + 4xy - e^x \cos(y)$.

c) Find an entire function $f(z)$ such that $Re(f) = u$. Your answer must be expressed as a function of $z = x + iy$, not x and y.

Answer: $f(z) = iz^3 + 2z^2 - ie^z$.

5. a) (6 points) Find the image of the vertical line $x = 2$ under the function $f(z) = e^{-z}$.

Answer: A general point on this line has the form $z = 2 + iy$ and $f(2 + iy) = e^{-2-2yi} = e^{-2}e^{-2yi}$ has absolute value e^{-2}. As y varies through all real numbers, e^{-2yi} varies through all points on the unit circle. Hence, the image is the circle of radious e^{-2}, centered at the origin.

b) (12 points) Let $\text{Log}(z)$ be the principal branch of the logarithm function defined and analytic on the open subset $\Omega := \{x + iy \text{ such that } y \neq 0 \text{ or } x > 0\}$ (the complex plane minus the set of non-poisitive real numbers). Find the set S of all z in Ω satisfying the equation $\text{Log}(z^4) = 4\text{Log}(z)$. Describe the conditions the equation imposes on the polar form of z and include a sketch of the set S.

Answer: Write $z = re^{i\theta}$, $r > 0$, $\theta = \text{Arg}(z) \in (-\pi, \pi)$. Then $z^4 = r^4 e^{i4\theta}$ and the equation $\text{Log}(z^4) = 4\text{Log}(z)$ becomes

$$\ln(r^4) + i\text{Arg}(e^{i4\theta}) = 4\ln(r) + i4\theta \qquad (3)$$

Now, $\ln(r^4) = 4\ln(r)$, for all $r > 0$. $\text{Arg}(e^{i4\theta}) = 4\theta + 2k\pi$, where k is the unique integer, such that $4\theta + 2k\pi$ belongs to $(-\pi, \pi)$. Equation (3) is thus equivalent to the equation $k = 0$, i.e., to the condition that 4θ belongs to $(-\pi, \pi)$. Summarizing, the set S is given, in polar form, by

$$S = \left\{ z = re^{i\theta} \text{ such that } r > 0 \text{ and } -\frac{\pi}{4} < \theta < \frac{\pi}{4} \right\}.$$

Geometrically, S is the region between the lines $y = x$ and $y = -x$, to the right of the y-axis.