Math 421 Midterm 1 Fall 2009

Name:____________________________

1. (36 points) Compute the following (in cartesian or polar form):

   a) Compute the polar form of $z = \dfrac{8}{-\sqrt{2}+\sqrt{2}i}$.

   b) $|z^{-2}|$, where z is given in part a.

   c) $\mathrm{Log}(a^3)$, where $a = 2e^{i[4\pi/5]}$.

d) Find all values of $(1-i)^{\frac{1}{4}}$. How many different values are there?

e) Find all values of $i^{[(1-i)/2]}$. How many different values are there?

2. (18 points) Determine which of the following functions is entire (analytic on the whole complex plane). Prove your answer. Carefully state each theorem you are using.

a) $f(z) = x^2 + y^2 + i(2xy)$.

b) $f(z) = e^{(x-y)} \sin(x+y) - ie^{(x-y)} \cos(x+y)$.

3. (10 points) Compute the Cartesian coordinates of $\cos\left(\frac{\pi}{4} - \frac{i}{2}\ln(2)\right)$. Show all your work and simplify your answer as much as possible.

4. (18 points) a) Prove that the function

$$u(x, y) \;=\; y^3 - 3x^2 y + 2x^2 - 2y^2 + e^x \sin(y)$$

is harmonic on the whole of $\mathbb{R}^2$.

b) Find a harmonic conjugate v of the function u.

c) Find an entire function $f(z)$ such that $Re(f) = u$. Your answer must be expressed as a function of $z = x + iy$, not x and y.

5. a) (6 points) Find the image of the vertical line $x = 2$ under the function $f(z) = e^{-z}$.

   b) (12 points) Let $\text{Log}(z)$ be the principal branch of the logarithm function defined and analytic on the open subset $\Omega := \{x + iy \text{ such that } y \neq 0 \text{ or } x > 0\}$ (the complex plane minus the set of non-poisitive real numbers). Find the set S of all z in Ω satisfying the equation $\text{Log}(z^4) = 4\text{Log}(z)$. Describe the conditions the equation imposes on the polar form of z and include a sketch of the set S.