Additional Problem

Let $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be the inverse of the function $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$. Let $U := \{z = x + iy \ : \ x > 0\}$ be the open right half-plane. Set

$$f(z) \quad := \quad \frac{\ln(x^2 + y^2)}{2} \ + \ i \arctan\left(\frac{y}{x}\right),$$

for $z = x + iy$ in U.

1. Show that $f(z) = Log(z)$, for all $z \in U$, where $Log(z)$ is the principal branch of $\log(z)$. Conclude that $f(z)$ is analytic.
2. Provide a second proof that $f(z)$ is analytic by checking that the assumptions of the Theorem in section 22 are satisfied (the partials of the real and imaginary components are continuous, and the Cauchy-Riemann equations are satisfied).
3. Use the function $\arccos : (-1, 1) \to (0, \pi)$ in order to express the imaginary part of $Log(z)$ in the upper-half-plane $\mathbb{H} := \{z = x + iy \ : \ y > 0\}$.