Solutions for Math 132 Fall '02 Final

1. We have $a_n = (\frac{n^2+2}{2n^2-1})^n$, whose nth root is simply $a_n^{1/n} = \frac{n^2+2}{2n^2-1}$. We consider the limit

$$\lim a_n^{1/n} = \lim \frac{n^2+2}{2n^2-1} = \lim \frac{1+2/n^2}{2-1/n^2} = 1/2.$$

Since this limit exists and is less than 1, the series converges by the nth root test.

2. Let $P(t)$ be the position (in meters) at time t, then $P'(t) = v(t)$ is the velocity in m/s at time t and $P''(t) = v'(t) = a(t)$ is the acceleration. Since $a(t) = -1$, we have $v(t) = -t + v_0$ where v_0 is some constant. Plugging in $t = 0$, we get $v(0) = 0 + v_0 = 4$, thus $v_0 = 4$ and

$$v(t) = -t + 4.$$

Note that $v(t) \geq 0$ for $0 \leq t \leq 4$ and $v(t) \leq 0$ for $4 \leq t \leq 6$. To get the total distance traveled, we must integrate the *absolute value* of the velocity.

(a) We get the distance traveled is

$$\begin{aligned}
\int_0^6 |-t+4|dt &= \int_0^4 (-t+4)dt + \int_4^6 (t-4)dt \\
&= (-t^2/2 + 4t)|_0^4 + (t^2/2 - 4t)|_4^6 \\
&= -8 + 16 + (18 - 24 - 8 + 16) = 10m.
\end{aligned}$$

(b) On the other hand, the total displacement from $t = 0$ to $t = 6$ is simpler to calculate:

$$\int_0^6 v(t)dt = \int_0^6 (-t+4)dt = -t^2/2 + 4t|_0^6 = -18 + 24 = 6m.$$

3. We have a water-holding parabola with lowest point at $(0, -4)$ and its reflection over the x-axis. The points of intersection are have x coordinate satisfying $4 - x^2 = x^2 - 4$ i.e. $x = \pm 2$. Thus the curves meet at $(2, 0)$ and $(-2, 0)$. The area between the curve is calculated by either of the integrals

$$\int_{-2}^2 [(4-x^2) - (x^2-4)]dx = 4\int_0^2 (4-x^2)dx.$$

However you slice it, the area between them is $4(4x - x^3/3)|_0^2 = 64/3$.

4. We rejoice at the sight of the odd power of $\sin(x)$ and immediately borrow one of these $\sin(x)$'s to couple with dx and put $u = \cos(x), du = -\sin(x)dx$. We

then have

$$\begin{aligned}
\int \sin^5(x)\cos^2(x)dx &= \int \sin^4(x)\cos^2(x)[\sin(x)dx] \\
&= \int (1-\cos^2(x))^2\cos^2(x)[\sin(x)dx] \\
&= \int -(1-u^2)^2u^2du \\
&= \int -(1-2u^2+u^4)u^2du \\
&= \int (-u^2+2u^4-u^6)du \\
&= -u^3/3+2u^5/5-u^7/7+C \\
&= -\cos^3(x)/3+2\cos^5(x)/5-\cos^7(x)/7+C.
\end{aligned}$$

Now we use the fundamental theorem to calculate the definite integral to have value

$$-\cos^3(x)/3+2\cos^5(x)/5-\cos^7(x)/7|_0^{\pi/2}=0!$$

(That's 0 (surprise!) which is equal to 0, not $0! = 0 factorial$ which everybody knows is equal to 1.)

5. For $x = t(t^2-3), y = 3(t^2-3)$, we need to calculate the slope of the tangent, which is, of course dy/dx. We use the chain rule to express this as $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Now

$$\frac{dx}{dt} = 1(t^2-3)+t(2t) = 3t^2-3, \qquad \frac{dy}{dt} = 6t.$$

Thus,

$$\frac{dy}{dx} = \frac{6t}{3(t^2-1)} = \frac{2t}{t^2-1}.$$

This quantity is 0, giving a horizontal tangent, when the numerator vanishes, i.e. for $t = 0$, or at the point $(0,-9)$. It is undefined (i.e. the denominator is 0), giving a vertical tangent when $t = \pm 1$ i.e. at the points $(2,-6)$ and $(-2,-6)$.

6. It is always a good idea to get acquainted with a series before you start investigating its convergence/divergence behavior. So, write out the first few (say 3) terms of the series to gain some familiarity with your opponent:

$$\sum_{n=0}^{\infty}\left(\frac{\ln x}{2}\right)^{2n} = \left(\frac{\ln x}{2}\right)^0+\left(\frac{\ln x}{2}\right)^2+\left(\frac{\ln x}{2}\right)^4+\cdots = 1+\left(\frac{\ln x}{2}\right)^2+\left(\frac{\ln x}{2}\right)^4+\cdots$$

First of all, let's note that this is a geometric series because it is of the type $\sum_{n=0}^{\infty}(BLOB)^n$. How come? Because $GOOP^{2n} = (GOOP^2)^n$. So for our series we have

$$\sum_{n=0}^{\infty}\left(\frac{\ln x}{2}\right)^{2n} = \sum_{n=0}^{\infty}\left[\left(\frac{\ln x}{2}\right)^2\right]^n = \sum_{n=0}^{\infty}(BLOB)^n$$

where

$$BLOB = \left(\frac{\ln x}{2}\right)^2.$$

Such a series converges exactly when $|BLOB| < 1$ and in that case it converges to $\frac{\text{first term}}{1-BLOB}$. Look carefully and notice that the series begins with $n = 0$ not $n = 1$ and $BLOB^0 = 1$ so the first term of the series is 1. [Of course we know this because we wrote out the first few terms.] So, summing up, we have the series converges exactly when

$$\begin{aligned}
\left|\left(\frac{\ln x}{2}\right)^2\right| &< 1 \text{ which means}\\
\left|\frac{\ln x}{2}\right| &< 1 \text{ which means}\\
|\ln x| &< 2 \text{ which means}\\
-2 < \ln x &< 2 \text{ which means}\\
e^{-2} < e^{\ln x} &< e^2 \text{ which means}\\
e^{-2} < x &< e^2.
\end{aligned}$$

For these values of x, our series converges to

$$\frac{1}{1-BLOB} = \frac{1}{1-(\ln x)^2/4} = \frac{4}{4-(\ln x)^2}.$$

7. To use the integral test, we just turn the Σ into a $\int$ and the n's into x's, tacking on a dx. The series converges if and only if the integral does.

a)

$$\begin{aligned}
\int_1^\infty \frac{1}{1+x^2}dx &= \lim_{B\to\infty}\int_1^B \frac{1}{1+x^2}dx\\
&= \lim_{B\to\infty} \arctan(x)|_1^B\\
&= \lim_{B\to\infty} \arctan(B) - arctan(1)\\
&= \pi/2 - \pi/4 = \pi/4.
\end{aligned}$$

Thus the series converges by the integral test.

b) First let's do the indefinite integral:

$$\int \frac{\ln x}{x}dx = \int udu = u^2/2+C = (\ln x)^2/2+C, \qquad \text{where } u = \ln x,\ du = dx/x.$$

Now on the improper integral:

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x}dx &= \lim_{B\to\infty}\int_1^B \frac{\ln x}{x}dx\\
&= \lim_{B\to\infty} (\ln x)^2/2|_1^B\\
&= \lim_{B\to\infty} (\ln B)^2/2 - (\ln 1)^2/2\\
&= \infty.
\end{aligned}$$

Hence the sum diverges.

8. Note the typo $n = 1$ instead of $i = 1$. We use ART (the absolute ratio test) of course. We have the nth term of the series is $a_n = (-1)^n 3^n (x-1)^n/n$, so $a_{n+1} = (-1)^{n+1}3^{n+1}(x-1)^{n+1}/(n+1)$. We want to calculate $L = \lim_{n\to\infty} |a_{n+1}/a_n|$. First we simplify this ratio:

$$\begin{aligned}
|a_{n+1}/a_n| &= \frac{3^{n+1}|x-1|^{n+1}n}{3^n|x-1|^n(n+1)}\\
&= 3|x-1|\frac{n}{n+1}.
\end{aligned}$$

Now it's immediate that $L = 3|x-1|$ since the ratio $n/(n+1)$ goes to 1. By the absolute ratio test, we know that

the series CONVERGES ABSOLUTELY when $L < 1$, i.e. when $|x-1| < 1/3$,

and we also know that the series DIVERGES when $L > 1$ i.e. when $|x-1| > 1/3$.

It remains to check what happens when $L = 1$ (that's when ART is inconclusive). $L = 1$ means $|x-1| = 1/3$, i.e. either $x - 1 = 1/3$ or $x - 1 = -1/3$, in other words it corresponds to $x = 4/3, 2/3$. When $x = 4/3$, the series is

$$\sum_{n=1} \infty \frac{(-1)^n 3^n (4/3-1)^n}{n} = \sum_{n=1} \infty \frac{(-1)^n}{n},$$

i.e. the series just becomes the alternating harmonic series, which converges by the Alternating Series Test. On the other hand, at the other endpoint, when $x = 2/3$, we just get the harmonic series, which diverges. Thus, the series converges absolutely for $2/3 < x < 4/3$, conditionally for $x = 4/3$ and diverges for all other x.

9. We know that

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + u^2/2! + u^3/3! + u^4/4! + ...$$

is a convergent power series for every u. Plugging in $u = -x^2$, we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + x^4/2! - x^6/3! + x^8/4! - ...$$

for every x. If we multipltiply this by x^2, it will still converge for every x giving

$$x^2 e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n!} = x^2 - x^4 + x^6/2! - x^8/3! + x^{10}/4! - ...$$

Now we have a theorem to the effect that we can integrate a power series term-by-term and that the resulting power series will converge in the same interval as the original power series. Thus we know that

$$\begin{aligned} \int x^2 e^{-x^2} dx &= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2n+2}}{n!} dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)} x^{2n+3} \\ &= C + x^3/3 - x^5/5 + x^7/(7 \cdot 2!) - x^9/(9 \cdot 3!) + x^{11}/(11 \cdot 4!) - ... \end{aligned}$$

is a convergent power series for every x.