1 Introduction

Let $R := \mathbb{C}[x]$ be the ring of polynomials. Let $V_{n,d}$ be the vector space of all $n\times n$ matrices with entries in R, such that the degree of each entry is $\leq d$. Clearly, $\dim(V_{n,d}) = n^2(d+1)$. Given a matrix $A = (a_{ij}(x))$ in $V_{n,d}$, its characteristic polynomial

$$\mathrm{char}_A(x,\lambda) \quad := \quad \det[A - \lambda I]$$

is a polynomial in two variables. The zero locus of $\mathrm{char}_A(x,\lambda)$ is an affine plane curve, called the *affine spectral curve of* A. Algebraic curves very often arise in other branches of mathematics as spectral curves (see [B2] for examples arising in classical mechanics). In problem 3 below you will prove the following statement, for all $d \geq 1$ and $n \geq 1$. Set $\mathbb{F}_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1})$ and let $p : \mathbb{F}_d \to \mathbb{P}^1$ be the natural morphism. Set $M := \mathcal{O}_{\mathbb{P}E}(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(d)$. Let M^n be the n-th tensor power of M.

Theorem 1 *There exists a Zariski dense open subset of $V_{n,d}$, consisting of matrices A, whose affine spectral curve is a Zariski open subset of a smooth connected projective curve $\widetilde{C}$ of genus $d\left(\frac{n(n-1)}{2}\right) - n + 1$. The curve $\widetilde{C}$ is naturally embedded*[1] *in the ruled surface $\mathbb{F}_d$ as a divisor in the linear system $|M^n|$.*

The construction introduces a morphism $char : V_{n,d} \to |M^n|$. In Problem 4 you will describe the fiber $char^{-1}(\widetilde{C})$ in terms of the spectral curve $\widetilde{C}$.

Set $F := \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}$. Key to the proof is the observation that an element A of $V_{n,d}$ corresponds to a homomorphism of $\mathcal{O}_{\mathbb{P}^1}$-modules $\varphi : F \to F \otimes \mathcal{O}_{\mathbb{P}^1}(d)$ as follows. Choose homogeneous coordinates (t_0, t_1) over $\mathbb{P}^1$. Set $\varphi_{ij}(t_0,t_1) := t_0^d a_{ij}(t_1/t_0)$. Then φ_{ij} is a homogeneous polynomial of degree d, hence a section of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. We get the isomorphism

$$\begin{aligned} V_{n,d} \quad &\cong \quad \mathrm{Hom}(F, F\otimes \mathcal{O}_{\mathbb{P}^1}(d)), \\ (a_{ij}) \quad &\mapsto \quad (\varphi_{ij}). \end{aligned}$$

N. Hitchin discovered in the 1980's that spectral curves play an important role in the study of n-dimensional irreducible complex representations of the fundamental group of a complex projective curve C of positive genus [H]. Hitchin's pairs (F,φ) consist of a rank n vector bundle F on C and its "endomorphism" $\varphi : F \to F \otimes \omega_C$ is twisted by the canonical line-bundle ω_C. Hitchin's spectral curves are embedded in the ruled surface $\mathbb{P}[\omega_C \otimes \mathcal{O}_C]$. The genus of Hitchin's spectral curve, which you will calculate below, is equal to half the dimension of the space of representations of the fundamental group.

Terminology: A rank n vector bundle over an algebraic variety X is a locally free $\mathcal{O}_X$-module of rank n. The following three objects are one and the same: a line-bundle, an invertible sheaf, and a locally free $\mathcal{O}_X$-module of rank 1.

[1] Note that the closure of such a curve in $\mathbb{P}^2$ has degree nd, so arithmetic genus $(nd-1)(nd-2)/2$. The latter is larger than the geometric genus by $n(d-1)[nd-2]/2$. Hence the closure in $\mathbb{P}^2$ is singular, except possibly when $d = 1$, or $(n,d) = (1,2)$.

2 Problems

1. Let C be a smooth curve, L a line bundle on C of degree d, $E := L \oplus \mathcal{O}_C$, and $p : \mathbb{P}E \to C$ the corresponding ruled surface. The line sub-bundle L of E corresponds to a section $\sigma_\infty : C \to \mathbb{P}E$, whose image is $\Sigma_\infty := \mathbb{P}L$. Let $\sigma_0 : C \to \mathbb{P}E$ be the section corresponding to the line sub-bundle $\mathcal{O}_C$ of E, and denote its image by Σ_0. The fiber of $[\mathbb{P}E \setminus \Sigma_\infty]$ over $y \in C$ can be naturally identified with the fiber $\overline{L}_y$ of L, and $\sigma_0(y)$ is its zero point. Simply associate to $\ell \in \overline{L}_y$ the point in $\mathbb{P}E$ corresponding to the line $\mathrm{span}_{\mathbb{C}}\{(\ell, 1)\}$ in the fiber of E.

   (a) Show that Σ_0 belongs to the linear system $|(p^*L) \otimes \mathcal{O}_{\mathbb{P}E}(1)|$ and Σ_∞ belongs to $|\mathcal{O}_{\mathbb{P}E}(1)|$. *Hint: Consider the tautological exact sequence*

   $$0 \to \mathcal{O}_{\mathbb{P}E}(-1) \to p^*(E) \to Q_{\mathbb{P}E} \to 0.$$

   Show that the section $(0, 1)$ of p^*E maps to a non-zero section of $Q_{\mathbb{P}E}$, which vanishes along Σ_0 with multiplicity 1. Then repeat your argument for the section $(1, 0)$ of $p^*(E \otimes L^{-1})$.

   (b) Let $D \subset \mathbb{P}E$ be an irreducible curve, which is disjoint from Σ_∞. Show that the class $[D]$ of D in $H^2(\mathbb{P}E, \mathbb{Z})$ is $n(df + h)$, where f is the class of the fiber, $h := c_1(\mathcal{O}_{\mathbb{P}E}(1))$, and $n := ([D], f)$. Conclude that the arithmetic genus of D is $g(D) = d\left(\frac{n(n-1)}{2}\right) + n[g(C) - 1] + 1$.
   Caution: In Proposition III.18 in Beauville's text [B1] his $\mathcal{O}_S(1)$ is our $Q_{\mathbb{P}E}$.

2. Keep the notation of problem 1. Set $M := (p^*L) \otimes \mathcal{O}_{\mathbb{P}E}(1)$. Following is an explicit construction of smooth curves in the linear system $|M^n|$, which are disjoint from Σ_∞. Choose $b_i \in H^0(C, L^i)$, $0 \le i \le n$. Set $b := (b_0, b_1, \ldots, b_n)$ and $a_i := p^*b_i$. Choose a section λ_1 of $H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1))$, with divisor Σ_∞ (λ_1 is unique, up to a scalar factor). If we identify $\mathcal{O}_{\mathbb{P}E}(1)$ with $\mathcal{O}_{\mathbb{P}E}(\Sigma_\infty)$, then λ_1 can be the section 1 of the latter. Choose a section λ_0 of $H^0(\mathbb{P}E, M)$, with divisor Σ_0. We get the section

   $$\sigma_b := \sum_{i=0}^{n} a_i \lambda_1^i \lambda_0^{n-i} \quad \in \quad H^0(\mathbb{P}E, M^n). \tag{1}$$

   Denote by $\widetilde{C}_b$ the divisor in $|M^n|$ corresponding to σ_b.

   (a) Show that if $b_0 \neq 0$, then the intersection $\widetilde{C}_b \cap \Sigma_\infty$ is empty.

   (b) Show that if $b_0 \neq 0$, $b_i = 0$, for $1 \le i \le n-1$, and the divisor of b_n in $|L^n|$ consists of nd distinct points of C, then the curve $\widetilde{C}_b$ is smooth and irreducible. *Note: Points in a linear system, corresponding to smooth divisors, form a Zariski open subset (see Hartshorne's Algebraic Geometry, Ch. I, section 5, Problem 5.15).*

   (c) Prove that $H^0(\mathbb{P}E, M^n)$ decomposes as the direct sum $\oplus_{i=0}^n \lambda_1^i \lambda_0^{n-i} p^* H^0(\mathbb{P}E, L^i)$. Conclude that every section of $H^0(\mathbb{P}E, M^n)$ is of the form given in Equation (1). Hint: It suffices to establish the direct sum decomposition

   $$H^0(\mathbb{P}E, M^k) = \lambda_0 H^0(\mathbb{P}E, M^{k-1}) \oplus \lambda_1^k p^* H^0(C, L^k),$$

for all $k \geq 1$. Note first the isomorphism $\sigma_0^*(M) \cong L$, and use it to construct the short exact sequence $0 \to M^{k-1} \xrightarrow{\lambda_0} M^k \longrightarrow (\sigma_0)_*(L^k) \to 0$.

3. **Construction of projective spectral curves:** Keep the notation of problems 1 and 2. Let F be a locally free coherent sheaf of rank n over C, $\varphi : F \to F \otimes L$ a homomorphism of $\mathcal{O}_C$-modules, and $p^*(\varphi) : p^*F \to p^*(F \otimes L)$ its pull-back to $\mathbb{P}E$. Set
$$\tilde{\varphi} \quad := \quad [p^*(\varphi) \otimes \lambda_1 - id_F \otimes \lambda_0] \quad : \quad p^*F \quad \longrightarrow \quad (p^*F) \otimes M. \qquad (2)$$
Then the determinant[2] $\det(\tilde{\varphi})$ is a section of M^n. The divisor $\widetilde{C} \in |M^n|$ of $\det(\tilde{\varphi})$ is called the **spectral curve** of φ.

   (a) Show that the spectral curve $\widetilde{C}$ of φ is disjoint from Σ_∞.

   (b) Set $F := \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}$. Let A be a matrix in $V_{n,d}$ and $\varphi : F \to F \otimes \mathcal{O}_{\mathbb{P}^1}(d)$ the associated homomorphism. Write $char_A(x, \lambda) = \sum_{i=0}^n c_i(x)\lambda^{n-i}$. Set $b_i := t_0^{di} c_i(t_1/t_0)$ and let $b = (b_0, \ldots, b_n)$. Show that the spectral curve of φ is equal to the curve $\widetilde{C}_b$ constructed in $\mathbb{F}_d := \mathbb{P}[\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}]$ in problem 2.

   (c) Let $char : V_{n,d} \to |M^n|$ be the morphism sending a matrix A to its spectral curve (a divisor in the linear system on $\mathbb{P}E$). Show that the image of the morphism $char$ contains the divisor of every curve considered in Question 2b.

   (d) Prove Theorem 1.

4. Keep the notation above.

   (a) Let $\tilde{g}$ be the genus of the generic spectral curve in Theorem 1. Verify the equality
$$\dim(V_{n,d}) \quad = \quad \tilde{g} + \dim|M^n| + \dim[PGL(n, \mathbb{C})].$$

   (b) The group $GL(n, \mathbb{C})$ acts on $V_{n,d}$ by conjugation, and the action factors through $PGL(n, \mathbb{C})$. Show that the morphism $char : V_{n,d} \to |M^n|$ is invariant under the $PGL(n, \mathbb{C})$-action.

   (c) Show that the co-kernel of the homomorphism $\tilde{\varphi}$, given in Equation (2), is an $\mathcal{O}_{\mathbb{P}E}$-module, whose set-theoretic support is the spectral curve $\widetilde{C}$. The sheaf $\widetilde{F} := \mathrm{coker}(\tilde{\varphi}) \otimes M^{-1}$ is a quotient of p^*F. $\widetilde{F}$ is called the **eigen-line-bundle** of φ. Prove the equality $\chi(\widetilde{F}) = \chi(F)$, where χ is the sheaf cohomology Euler characteristic (on $\mathbb{P}E$ and on C).

   (d) Recall that $p_*(p^*F) \cong F \otimes (p_*\mathcal{O}_{\widetilde{C}})$, by the projection formula. Let $q : p^*F \to \widetilde{F}$ be the quotient homomorphism. Prove that the composition
$$F \xrightarrow{id_F \otimes 1} F \otimes p_*(\mathcal{O}_{\widetilde{C}}) \cong p_*(p^*F) \xrightarrow{p_*(q)} p_*\widetilde{F}$$

[2] If $F = \oplus_{i=1}^n \mathcal{O}_C$ is the trivial vector bundle, then $\tilde{\varphi}$ is an $n \times n$ matrix, whose entries are sections of M. The determinant $\det(\tilde{\varphi})$ is then the usual determinant, where we replace the product of n entries by their tensor product. For a general F, the homomorphism $\tilde{\varphi}$ induces a homomorphism
$$\overset{n}{\wedge} \tilde{\varphi} \; : \; \overset{n}{\wedge} (p^*F) \quad \longrightarrow \quad \overset{n}{\wedge} [(p^*F) \otimes M] \;\cong\; [\overset{n}{\wedge} (p^*F)] \otimes M^n.$$
It corresponds to a section $\det(\tilde{\varphi})$ of M^n, since $\overset{n}{\wedge} (p^*F)$ is an invertible sheaf.

is an isomorphism. Hint: It suffices to prove injectivity, by part 4c. See Remark 2 for the meaning of this isomorphism.

Remark 2 When $\widetilde{C}$ is smooth, the sheaf $\widetilde{F}$ is a locally free $\mathcal{O}_{\widetilde{C}}$-module of rank 1, by part 4d. The isomorphism class of $\widetilde{F}$ determines the isomorphism class of the pair (F, φ), and so the $PGL(n, \mathbb{C})$-orbit of the matrix $A \in V_{n,d}$, as follows. Let $\mu : \widetilde{F} \to \widetilde{F} \otimes M$ be the homomorphism, given by tensoring with the section λ_0 of M. The push-forward $p_*(\mu)$ is equal[3] to the homomorphism $\varphi : F \to F \otimes L$, up to conjugation of φ by an automorphism of F. Set $\tilde{d} := \chi(\widetilde{F}) + 1 - \tilde{g}$. The algebraic variety $\mathrm{Pic}^{\tilde{d}}(\widetilde{C})$, of degree $\tilde{d}$ line-bundles on $\widetilde{C}$, is a $\tilde{g}$-dimenstional smooth algebraic variety (Its dimension is equal to $h^1(C, \mathcal{O}_C)$, by the discussion in Section I.10 of Beauville's text on the exponential sequence [B1]). Hence, the fiber $char^{-1}(\widetilde{C})$ is an algebraic subset of $V_{n,d}$ of dimension at most $\widetilde{g} + \dim PGL(n, \mathbb{C})$. This must be exacltly the dimension of the fiber, by part 4a. See [BNR] for a detailed exposition.

5. Do problems 1, 2, 5, 6 in Chapter III page 37 of Beauville's text [B1].

References

[B1] **Beauville, A.:** *Complex Algebraic Surfaces.* Second Edition. London Math. Soc. Student Texts 34, Cambridge Univ. Press 1996.

[B2] **Beauville, A.:** *Jacobiennes des courbes spectrales et systèmes hamiltoniens complètement intégrables.* Acta Math. 164, 211-235 (1990)

[BNR] **Beauville, A., Narasimhan, M. S., Ramanan, S.:** *Spectral curves and the generalized theta divisor.* J. Reine Angew. Math. 398, 169-179 (1989)

[H] **Hitchin, N.J.:** *The self-duality equations on a Riemann surface.* Proc. Lond. Math. Soc. **55** (1987) 59–126.

[3] The above statement is due to the fact that a fiber of $\widetilde{F}$ over a point x of $\widetilde{C}$ is naturally identified with the x-eigen-line of the fiber $\overline{F}_{p(x)}$ of F over $p(x)$, provided the eigenvalue x has multiplicity one (i.e., provided x is not a ramification point of $\widetilde{C} \to C$). Furthermore, μ acts on this fiber of $\widetilde{F}$ via tensorization with the corresponding eigenvalue $x \in \overline{L}_{p(x)}$. Finally, the fiber of $p_*\widetilde{F}$ over $y \in C$ is naturally identified with the direct sum of the fibers of $\widetilde{F}$, over points in $p^{-1}(y)$, provided y is not a branch points of $\widetilde{C} \to C$.